18.085 Computational Science and Engineering	March 29, 2010
Answers to Problem Set 6	
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3.1.3. Integrating from 0 to 1, $\int_0^1 f(x)dx = \int_0^1 -\frac{dw}{dx}dx = -(w(1)-w(0)) = 0$, which is the sought condition on f.

3.1.4. Integrating $-u" = e^x$ twice, we obtain $u = -e^x + Ax + B$. Now $u(0) = u(1) = 0$ imply $A = e-1, B = 1$ i.e. $u = -e^x + (e-1)x + 1$.

3.1.7. For $p = 0$, the equation becomes $-u" = 0$, whose general solution is $u = Ax+B$. For $p \neq 0$, setting $u = e^{\lambda x}$ and solving for λ we get $\lambda = 0$ or $\lambda = p$. Thus the general solution in this case is $u = Ae^{px} + B$.

3.1.10. The discrete equation for $-u" = 2, u(0) = u(1) = 0$ has the form $K\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = F$, where $F_i = \int_0^1 2\phi_i dx = \frac{1}{2}$ (twice the area of the graph under ϕ_i) and $K_{ij} = \int_0^1 \phi_i^{'}\phi_j^{'}dx$. Since ϕ_2, ϕ_3 are just shifts of ϕ_1, we only need to compute $K_{11} = \int_0^{1/2} 16dx = 8, K_{12} = \int_{1/4}^{1/2}(-4)(4)dx = -4$, hence the equation becomes $\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$. These values perfectly match the values of the actual solution $u = x - x^2$ at $x = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}$.

3.1.12. We will use the same ϕ_i as in the previous problem. Again, because of symmetry, we only need to compute $M_{11} = \int_0^{1/4}(4x)^2dx + \int_{1/4}^{1/2}(1-4x)^2dx = \frac{1}{6}, M_{12} = \int_{1/4}^{1/2}(4x)(1-4x)dx = \frac{1}{24}$. The desired matrix is

$$M = \begin{bmatrix} \frac{1}{6} & \frac{1}{24} & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ 0 & \frac{1}{24} & \frac{1}{6} \end{bmatrix}.$$

3.1.14. Integrating $\frac{d}{dx}(w(x)v(x)) = w(x)\frac{dv}{dx} + v(x)\frac{dw}{dx}$ from 0 to 1, and setting $w(x) = c(x)\frac{du}{dx}$, we obtain $\left[c(x)v(x)\frac{du}{dx}\right]_0^1 = \int_0^1 d\left(c(x)\frac{du}{dx}\right)v(x)dx + \int_0^1 c(x)\frac{du}{dx}\frac{dv}{dx}dx$, i.e. equation (21).

3.1.15. Once finds $\phi_5 = -\frac{4}{h^2}(x-h)(x-2h), \phi_5^{'} = -\frac{4}{h^2}(2x-3h)$. Using Simpson's formula, $K_{55} = \frac{h}{6}\left(\phi_5^{'}(h)\right)^2 + 0 + \frac{h}{6}\left(\phi_5^{'}(2h)\right)^2 = \frac{16}{3h}$.