18.085 HOMEWORK 4 SOLUTIONS

2.2.7. Computing numerically, $\lambda^{32} = 0.9997985 - 0.02006909i, \theta = 0.02007\text{rad}$. The 2 Taylor series centered at $h = 0$ are given by $\lambda = \frac{1+ih/2}{1-ih/2} = (1 + \frac{ih}{2})(1 - \frac{ih}{2} + \frac{h^2}{4} - \frac{ih^3}{8} + ...) = 1 + ih - \frac{1}{2}h^2 - \frac{1}{4}ih^3 + ...$, and $e^{ih} = 1 + ih + \frac{(ih)^2}{2!} + \frac{(ih)^3}{3!} + ... = 1 + ih - \frac{1}{2}h^2 - \frac{i}{6}h^3 + ...$ Thus the 2 series differ in the 3^{rd} power of h.

2.3.3. Method 5 should yield the most accurate solutions.

2.3.4. We have $a_1 = [2\ 2\ 1]^T \Rightarrow q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{3}[2\ 2\ 1]^T$. Next, $b_2 = a_2 - (q_1, a_2)q_1 = [-1\ 2\ 2]^T - \frac{4}{9}[2\ 2\ 1]^T = \frac{1}{9}[-17\ 10\ 14]^T$, and $q_2 = \frac{b_2}{\|b_2\|} = \frac{[-17\ 10\ 14]^T}{3\sqrt{65}} \Rightarrow R = [q_1\ q_2]^T[a_1\ a_2] = \begin{bmatrix} 3 & 4/3 \\ 0 & \sqrt{65}/3 \end{bmatrix}$.

2.3.10. We are trying to solve $C = 4, C = 1, C = 0, C = 1$, i.e. $A[C] = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}, A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Hence $A^TA[C] = A^T \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow 4C = 6 \Rightarrow C = \frac{3}{2}$. Thus the best horizontal line is $y = \frac{3}{2}$.

2.4.2. We have $A_{triangle} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, and its nullspace is $\{\begin{bmatrix} C \\ C \\ C \end{bmatrix}\}$ because the underlining graph is connected. $A^T_{triangle} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. The 3 equations of $A^T_{triangle}x = 0$ are $x_1 + x_3 = 0, x_1 - x_2 = 0, x_2 + x_3 = 0$ hence $N(A^T_{triangle}) = \{\begin{bmatrix} C \\ C \\ -C \end{bmatrix}\}$.

2.4.5. If there is an edge $i \to j$ in the graph, then some row of A has the form $[0...0\ -1...0\ 1...0]$ with -1 and 1 in positions i and j respectively. Then the corresponding equation of the system $Au = 0$ reads $-u_i + u_j = 0 \Rightarrow u_i = u$. For any $i \neq j$, since the graph is connected, there exists a path $i \to i_1 \to ... \to i_l \to j$. By the earlier argument, $u_i = u_{i_1}, u_{i_1} = u_{i_2}, ..., u_{i_l} = u_j$, hence $u_i = u_j$. Therefore, the solutions to $Au = 0$ are $\begin{bmatrix} C \\ \vdots \\ C \end{bmatrix}$.

2.4.6. The ii entry of A^TA is $\sum_{k=1}^n a_{ki}a_{ki} = \sum_{k-\text{adjacent to } i}(\pm 1)^2 = \deg i$. Hence $tr(A^TA) = \sum_{i=1}^n \deg i = 2m$, because each edge is counted twice.

2.4.7. We have $A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, hence

$$K = A^T \begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix} A = \begin{bmatrix} c_1 & 0 & 0 & -c_1 \\ 0 & c_2 & 0 & -c_2 \\ 0 & 0 & c_3 & -c_3 \\ -c_1 & -c_2 & -c_3 & c_1+c_2+c_3 \end{bmatrix}$$

Grounding node 4 means deleting the 4th column of A, hence deleting the last row and column of K. Thus, $K_{red} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$, $\det K = c_1 c_2 c_3 (c_1 + c_2 + c_3)$.

2.4.9. We have $A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$, $A^T A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$.

Removing row 5 and column 5 of $A^T A$ gives $(A^T A)_{red} = K = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} =$ T_4, hence $K^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. Using Matlab, one finds the eigenvalues of K to be $1, 0.1206, 2.3473, 3.5321$, and $\det K = 1$.