HOMEWORK 3 SOLUTIONS

1.6.15. The upper left determinants for $A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$ are $c, c^2 - 1$ and $c^3 - 3c + 2 = (c-1)^2(c+2)$. They are all positive iff $c > 1$. The upper left determinants for $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}$ are $1, d - 4$ and $\det B = 5d + 48 - 9d - 36 = 4(3 - d)$. They cannot be all positive.

1.6.16. The eigenvalues of $A^{-1}$are the inverses of the eigenvalues of A, hence are also positive. Second proof: $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is positive definite iff $a > 0, ac - b^2 > 0$. In particular, $c > 0$. Now $A^{-1} = \frac{1}{ac-b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$ has determinant $\frac{1}{ac-b^2} > 0$ and top left entry $c > 0$, hence is positive definite.

1.6.17. $[u_1\ u_2\ u_3] \begin{bmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 4u_1^2 + 5u_3^2 + 2u_1u_3 + 4u_2u_3 + 2u_1u_2$. Since there is no u_2^2 term, setting $u_1 = u_3 = 0$ makes $u^TAu = 0$.

1.6.21. For $f_1 = \frac{1}{4}x^4 + x^2y + y^2$, we have $\frac{\partial f_1}{\partial x} = x(x^2+2y)$, $\frac{\partial f_1}{\partial y} = x^2+2y$, so setting both partials equal to 0, we get a curve of critical points: $x^2+2y = 0$. On this curve, the Hessian is $\begin{bmatrix} 3x^2 + 2y & 2x \\ 2x & 2 \end{bmatrix}$, with top entry $2x^2 + (x^2 + 2y) = 2x^2 \geq 0$ and determinant $2(x^2 + 2y) = 0$. Thus the Hessian is positive semidefinite at all critical points, so we have minima. For $f_2 = x^3 + xy - x$, $(f_2)_x = 3x^2 + y - 1, (f_2)_y = x$. Setting them both 0, we find the only critical point $(0, 1)$. The Hessian is $\begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix}_{(0,1)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, with negative determinant, hence we have a saddle point.

1.6.23. The Hessian for $f(x, y) = 4x^2 + 12xy + cy^2$ is $\begin{bmatrix} 8 & 12 \\ 12 & 2c \end{bmatrix}$, with positive top left entry and determinant $16(c - 9)$. Hence, for $c > 9$, the Hessian is positive definite and graph of f is a bowl, for $c = 9$ it's positive semidef, so we have a trough (minimal line is $2x + 3y = 0$) and for $c < 9$ is indefinite so we have a saddle.

2.1.4. In the free-free case by (11) on page 106, $A^TCA = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}$.
If we sum the 3 rows we get $[0\ 0\ 0]$, hence summing the 3 equations of $A^TCAu = f$ gives $0 = f_1+f_2+f_3$. Now we solve $A^TCAu = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \to \left[\begin{array}{ccc|c} c_2 & -c_2 & 0 & -1 \\ -c_2 & c_2 + c_3 & -c_3 & 0 \\ 0 & -c_3 & c_3 & 1 \end{array}\right] \to$

$\left[\begin{array}{ccccc} c_2 & -c_2 & 0 & | & -1 \\ 0 & c_3 & -c_3 & | & -1 \\ 0 & 0 & 0 & & 0 \end{array}\right]$, hence u_3 may be chosen as a free variable and the general solution is $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{c_2}(c_2u_2 - 1) \\ \frac{1}{c_3}(c_3u_3 - 1) \\ u_3 \end{bmatrix} = \begin{bmatrix} u_3 - \frac{1}{c_2} - \frac{1}{c_3} \\ u_3 - \frac{1}{c_3} \\ u_3 \end{bmatrix}$.

2.1.5. For simplicity, we assume $c_i = c$ and that the 3 masses are equal (to m). Then the gravitational forces are all mg so we have the equation

$$A^TCAu = \begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} \Leftrightarrow \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{bmatrix} u = mg \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Leftrightarrow K_3u = \frac{mg}{c} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

.Hence the displacements are given by $u = \frac{mg}{c} \cdot \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{mg}{2c} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$.

Next, we find the forces: $w = CAu = c \begin{bmatrix} u_1 \\ u_2 - u_1 \\ u_3 - u_2 \\ -u_3 \end{bmatrix} = \frac{mg}{2} \begin{bmatrix} 3 \\ 1 \\ -1 \\ -3 \end{bmatrix}$. So the reaction forces on the top and bottom are $\frac{3}{2}mg$ and $-\frac{3}{2}mg$ respectively. In absolute value, they sum to the total gravitational force $3mg$.

2.1.6. In the fixed-free case, $K = A^TCA = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} = \begin{bmatrix} c_2+1 & -c_2 & 0 \\ -c_2 & c_2+1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. For $c_2 = 10$, $u = K^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.2 \\ 4.2 \end{bmatrix}$ and for $c_2 = 100, u = K^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.02 \\ 4.02 \end{bmatrix}$. Basically, when we increase c_2, m_1 and m_2 behave like a single mass.

2.2.5. (a) $\frac{d}{dt} \|u(t)\|^2 = 2(u_1u_1' + u_2u_2' + u_3u_3') = 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2) = 0$. Having 0 derivative, $\|u(t)\|^2$ is constant, so $\|u(t)\|^2 = \|u(0)\|^2$.
(b) For any n we have $(A^n)^T = (A^T)^n$, hence $Q^T = \left(I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + ...\right)^T = I + tA^T + \frac{t^2}{2!}(A^T)^2 + \frac{t^3}{3!}(A^T)^3 + ... = e^{tA^T} = e^{-At}$. Hence $QQ^T = e^{At}e^{-At} = I$, where the last equality follows from the formal power series identity $e^xe^{-x} = 1$.