Solutions to Problem Set 2 #

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[1.3-5]
2 and 7

[1.3-6]
15, 5 and 10

[1.3-7]
Check by direct matrix multiplication.

\[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]

Therefore, the answer is 1 and -1. Then by multiplication, we get vectors \((1, 1), (2, 1), (3, 2), (5, 3), (8, 5)\), which give us the sequence

\[1, 1, 2, 3, 5, 8, \ldots\]

[1.4-2]
Let \(u(x) = Ax + B\) when \(0 \leq x \leq a\), and \(u(x) = Cx + D\), when \(a \leq x \leq 1\). By the boundary conditions, \(u'(0) = 0, u(1) = 4\). Together with the jump condition and slope change, we have

\[A = 0, B = 5 - a, C = -1, D = 5.\]

[1.4-6]
Let \(u(x) = Ax + B\) when \(0 \leq x \leq a\), and \(u(x) = Cx + D\), when \(a \leq x \leq 1\). By the periodic conditions, we have \(u'(0) = u'(1)\) and \(u(0) = u(1)\). By the continuity at \(a\), we have \(Aa + B = Ca + D\). But then the change in slope at \(x = a\) will give us \(C - A = -1\), which contradicts \(C = A\) that we get from the boundary conditions.

[1.4-12]
Let \(u(x) = Dx^3 + Ex^2 + Fx + G + C(x)\), where \(C(x) = \frac{1}{6}x^3\) for \(x > 0\). Then by the condition that \(u(-1) = 0, u''(-1) = 0, u(1) = 0\) and \(u''(1) = 0\), we have

\[D = -\frac{1}{12}, E = -\frac{1}{4}, F = 0, G = \frac{1}{6}.\]

[1.4-15]
\[ \int_{-\infty}^{+\infty} g(x)\delta'(x)dx = g(x)\delta(x)|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \delta(x)dg(x) = 0 - \int_{-\infty}^{+\infty} \delta(x)g'(x)dx = -g'(0) \]

[1.5-2]
\[ \lambda = 2(1 - \cos \pi h) \]

[1.5-10]
For \( A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \), we have
\[ A = SAS^{-1} = \begin{pmatrix} 1 & \frac{1}{2^2} \\ 0 & \frac{1}{2^2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & \frac{1}{2^2} \end{pmatrix}. \]

Then check that \( A^2 = S\Lambda^2S^{-1} \) is direct by matrix multiplication.

Similar for \( A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \), we have
\[ A = SAS^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & \frac{-1}{5} \\ \frac{-1}{5} & \frac{1}{5} \end{pmatrix}. \]

Also, we have \( A^2 = S\Lambda^2S^{-1} \).

[1.5-11]
\[ A^{-1} = (SAS^{-1})^{-1} = (S^{-1})^{-1}(\Lambda)^{-1}S^{-1} = SAS^{-1}. \]

Also for \( A^3 \), we have \( A^3 = S\Lambda^3S^{-1} \).

[1.5-12]
\[ S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}. \] Then
\[ A = SAS^{-1} = \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix} \]

[1.6-12]
For \( A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \), we have \( A^TA \) positive definite because the columns of \( A \) are linearly independent.
For $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}$, we have $A^T A$ positive definite because the columns of $A$ are linearly independent.

For $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$, we have $A^T A$ not positive definite because the columns of $A$ are linearly dependent.