18.085 Computational Science and Engineering	April 26, 2010
Answers to Problem Set 10	
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4.2.1.a) We have

$$\begin{aligned} c_{mn} &= \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(x,y)e^{-imx}e^{-iny}dxdy \\ &= \frac{1}{4\pi^2}\int_0^{\pi}\int_0^{\pi} e^{-imx}e^{-iny}dxdy = \frac{1}{4\pi^2}\int_0^{\pi} e^{-imx}dx \int_0^{\pi} e^{-iny}dy. \end{aligned}$$

Since $\int_0^\pi e^{-imx}dx$ equals π if $m=0$, and $\frac{1-e^{-im\pi}}{im}$ otherwise, we get $c_{00} = 1/4, c_{m0} = c_{0m} = \begin{cases} 1/(2\pi im) & if\ m\ odd \\ 0 & otherwise \end{cases}$ for $m \neq 0$, and $c_{mn} = \begin{cases} \frac{-1}{\pi^2 mn} & if\ m,n\ odd \\ 0 & otherwise \end{cases}$ for $m,n \neq 0$.

b) $c_{mn} = \frac{1}{4\pi^2}\left(\int_0^\pi\int_0^\pi e^{-imx}e^{-iny}dxdy + \int_{-\pi}^0\int_{-\pi}^0 e^{-imx}e^{-iny}dxdy\right)$. Similarly to a), we obtain $c_{00} = 1/2, c_{m0} = c_{0m} = 0$ for $m \neq 0$ and $c_{mn} = \begin{cases} \frac{-2}{\pi^2 mn} & if\ m,n\ odd \\ 0 & otherwise \end{cases}$ for $m,n \neq 0$.

4.2.2. Since $\sin mx$ and $\sin ny$ are odd in x and y respectively, $S(x,y)$ will have a double sine series $\sum\sum b_{mn}\sin mx \sin ny$ if $S(x,y)$ is odd in x and y: $-S(x,y) = S(-x,y) = S(x,-y)$. The double sine functions are orthogonal:

$$\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(\sin kx \sin ly)(\sin mx \sin ny)dxdy = \begin{cases} 1 & if\ k=m\ and\ l=n \\ 0 & otherwise \end{cases}.$$

4.2.3. Using the orthogonality from the previous problem,

$$b_{mn} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} S(x,y)\sin mx \sin ny dxdy.$$

4.2.7. From the (1-dimensional) Fourier series of the delta function we obtain $\delta(x+y) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{-im(x+y)} = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{-imx}e^{-imy}$.

4.2.8. $x^4 = \frac{1}{8}(8x^4 - 8x^2 + 1) + \frac{1}{2}(2x^2-1) + \frac{3}{8} = \frac{1}{8}T_4 + \frac{1}{2}T_2 + \frac{3}{8}T_0$.

4.2.10. Let $B_n(x) = \begin{pmatrix} x & -1 & ... & & \\ -1 & 2x & -1 & & \\ & & & & \\ & -1 & \ddots & -1 \\ & & -1 & 2x \end{pmatrix}$ (n by n matrix). We compute its determinant by expanding along the *last* row:

$$\det B_n = 2x\det B_{n-1} - (-1)\det\begin{pmatrix} B_{n-2} & 0 \\ 0 \ ... -1 & -1 \end{pmatrix} = 2x\det B_{n-1} - \det B_{n-2}.$$

Thus $\det B_n$ is a polynomial in x satisfying the same recurrence as T_n. Since $\det B_1 = x = T_1, \det B_2 = 2x^2 - 1 = T_2$, it follows that $\det B_n = T_n$ for all n.

4.2.20. We have

$$\begin{aligned}(pu'_m)' + (q + \lambda_m w)u_m &= 0 \\ (pu'_n)' + (q + \lambda_n w)u_n &= 0\end{aligned}$$

Multiplying the first equation by u_n,the second by u_m and then subtracting, we obtain $u_n(pu'_m)' - u_m(pu'_n)' + (\lambda_m - \lambda_n)wu_m u_n = 0$. Integrating, $\int_a^b (\lambda_m - \lambda_n)wu_m u_n dx = \int_a^b u_n(pu'_m)' - u_m(pu'_n)'dx = [u_n(pu'_m) - u_m(pu'_n)]_a^b = 0$, by the boundary conditions. Since $\lambda_m \neq \lambda_n$, we get $\int_a^b wu_m u_n dx = 0$.

4.2.21. Bessel's equation: $r^2B'' + rB' + \lambda r^2 B = n^2 B \Rightarrow rB" + B' + (\lambda r - n^2/r)B = 0 \Leftrightarrow (rB')' + (\lambda r - n^2/r)B = 0$ which is a Sturm-Liouville equation with $u = B, p = r, q = -n^2/r, w = r$. Alternatively, one can try $u = B(r/2)$ and $p = r^2$. Legendre's equation: $(1-x^2)P" - 2xP' + \lambda P = 0 \Leftrightarrow ((1-x^2)P')' + \lambda P = 0$ is a Sturm-Liouville equation with $u = P, p = 1 - x^2, w = 1, q = 0$.