4.2.1. a) We have
\[
c_{mn} = \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(x, y)e^{-imx}e^{-iny}dxdy
\]
\[
= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-imx}e^{-iny}dxdy = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} e^{-imx}dx \int_{-\pi}^{\pi} e^{-iny}dy.
\]
Since \(\int_{0}^{\pi} e^{-imx}dx\) equals \(\pi\) if \(m = 0\), and \(\frac{1 - e^{-im\pi}}{im}\) otherwise, we get \(c_{00} = 1/4\), \(c_{m0} = c_{0m} = 0\) for \(m \neq 0\) and \(c_{mn} = \left\{ \begin{array}{ll} 1/(2\pi im) & \text{if } m \text{ odd} \\ 0 & \text{otherwise for } m \neq 0 \end{array} \right. \) and \(c_{mn} = \left\{ \begin{array}{ll} \frac{-1}{m\pi} & \text{if } m, n \text{ odd} \\ 0 & \text{otherwise for } m, n \neq 0 \end{array} \right. \)

b) \(c_{mn} = \frac{1}{4\pi^2} \left( \int_{0}^{\pi} \int_{0}^{\pi} e^{-imx}e^{-iny}dxdy + \int_{-\pi}^{0} \int_{-\pi}^{0} e^{-imx}e^{-iny}dxdy \right)\). Similarly to a), we obtain \(c_{00} = 1/2, c_{m0} = c_{0m} = 0\) for \(m \neq 0\) and \(c_{mn} = \left\{ \begin{array}{ll} \frac{-1}{m\pi} & \text{if } m, n \text{ odd} \\ 0 & \text{otherwise for } m, n \neq 0 \end{array} \right. \)

4.2.2. Since \(\sin mx\) and \(\sin ny\) are odd in \(x\) and \(y\) respectively, \(S(x, y)\) will have a double sine series \(\sum \sum b_{mn}\sin mx\sin ny\) if \(S(x, y)\) is odd in \(x\) and \(y\); \(-S(x, y) = S(-x, y) = S(x, -y)\). The double sine functions are orthogonal:
\[
\frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (\sin kx\sin ly)(\sin mx\sin ny)dxdy = \left\{ \begin{array}{ll} 1 & \text{if } k = m \text{ and } l = n \\ 0 & \text{otherwise} \end{array} \right.
\]

4.2.3. Using the orthogonality from the previous problem,
\[
b_{mn} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} S(x, y)\sin mx\sin nydxdy.
\]

4.2.7. From the (1-dimensional) Fourier series of the delta function we obtain
\(\delta(x + y) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{-im(x + y)} = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{-imx}e^{-imy}\).

4.2.8. \(x^2 = \frac{1}{8}(8x^4 - 8x^2 + 1) + \frac{1}{2}(2x^2 - 1) + \frac{3}{8} = \frac{1}{2}T_4 + \frac{1}{2}T_2 + \frac{3}{8}T_0\).

4.2.10. Let \(B_n(x) = \begin{pmatrix} x & -1 & \cdots & -1 \\ -1 & 2x & -1 & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ -1 & \cdots & -1 & 2x \end{pmatrix}\) \((n\ by\ n\ matrix)\). We compute its determinant by expanding along the last row:
\[
det B_n = 2x \det B_{n-1} - (-1)^{n-1} \det \begin{pmatrix} B_{n-2} & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 \\ \vdots & \ddots & \ddots & \ddots \\ -1 & \cdots & -1 & 2x \end{pmatrix} = 2x \det B_{n-1} - \det B_{n-2}.
\]

Thus \(\det B_n\) is a polynomial in \(x\) satisfying the same recurrence as \(T_n\). Since \(\det B_1 = x = T_1, \det B_2 = 2x^2 - 1 = T_2\), it follows that \(\det B_n = T_n\) for all \(n\).

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4.2.20. We have

\[(pu_m')' + (q + \lambda m w)u_m = 0\]
\[(pu_n')' + (q + \lambda n w)u_n = 0\]

Multiplying the first equation by \(u_n\), the second by \(u_m\) and then subtracting, we obtain \(u_n(pu_m')' - u_m(pu_n')' + (\lambda_m - \lambda_n)wu_mu_n = 0\). Integrating, \(\int_a^b (\lambda_m - \lambda_n)wu_mu_n\,dx = \int_a^b u_n(pu_m')' - u_m(pu_n')'\,dx = [u_n(pu_m') - u_m(pu_n')]_a^b = 0\), by the boundary conditions. Since \(\lambda_m \neq \lambda_n\), we get \(\int_a^b wu_mu_n\,dx = 0\).

4.2.21. Bessel's equation: \(r^2 B'' + r B' + \lambda r^2 B = n^2 B \Rightarrow r B'' + B' + (\lambda r - n^2/r)B = 0\) which is a Sturm-Liouville equation with \(u = B, p = r, q = -n^2/r, w = r\). Alternatively, one can try \(u = B(r/2)\) and \(p = r^2\). Legendre's equation: \((1 - x^2)P'' - 2xP' + \lambda P = 0 \Leftrightarrow ((1 - x^2)P')' + \lambda P = 0\) is a Sturm-Liouville equation with \(u = P, p = 1 - x^2, w = 1, q = 0\).