18.085 HOMEWORK 1 SOLUTIONS (SKETCH)

1.1.1. From $T_3^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, T_4^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ it's easy to guess the general pattern. The ij entry of T_n^{-1} is $n+1-\max(i,j)$.

1.1.7. We have $T_3^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, K_3^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ so $T_3^{-1} - K_3^{-1} = \frac{1}{4}\begin{bmatrix} 9 & 6 & 3 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$.

1.1.10. On one hand, $H = JTJ \Rightarrow$

$$H = J^{-1}T^{-1}J^{-1} = JT^{-1}J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

On the other hand, $H = UU^T, U = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow U^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, H^{-1} = (U^{-1})^T U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

1.1.12. Do row elimination on C_4: $\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 3/2 & -1 & -1/2 \\ 0 & -1 & 2 & -1 \\ 0 & -1/2 & -1 & 3/2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 3/2 & -1 & -1/2 \\ 0 & 0 & 4/3 & -4/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$. The last row of U is 0, hence $C_4$is singular. New non-zeros appear because of the -1 in the top right corner of C_4.

1.2.3. For $u = x^3$, $\frac{u(x+h)-u(x-h)}{2h} = \frac{((x+h)-(x-h))((x+h)^2+(x+h)(x-h)+(x-h)^2)}{2h} = 3x^2 + h^2$.

For $u = x^4$, $\frac{u(x+h)-u(x-h)}{2h} = \frac{((x+h)-(x-h))((x+h)+(x-h))((x+h)^2+(x-h)^2)}{2h} = 4x^3 + 4xh^2$. In each case the error term is exactly $\frac{1}{6}u'''(x)h^2$.

1.2.4. One directly checks that the inverse of $\Delta_- = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & & \\ & & & \cdots \end{bmatrix}$ is the sum matrix $U = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$. Since $\Delta_+ = \begin{bmatrix} -1 & 1 & \\ & -1 & 1 \\ & & \ddots \end{bmatrix}$, we obtain $\Delta_0 =$

$\frac{1}{2}(\Delta_+ + \Delta_-) = \frac{1}{2}\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & & \\ & & & \ddots & 1 \\ & & & -1 & 0 \end{bmatrix}$. For $n = 3, \Delta_0 = \frac{1}{2}\begin{bmatrix} & 1 & \\ -1 & & 1 \\ & -1 & \end{bmatrix}$,
which is singular since rows 1,3 sum to 0. For $n = 5$, Δ_0 is also singular since rows 1,3,5 sum to 0.

1.2.5b. By using Taylor expansion, we have

$$\begin{aligned} u(x+2h) &= u(x) + 2hu'(x) + \frac{1}{2}4h^2u''(x) + \frac{1}{6}8h^3u'''(x) + \frac{1}{24}16h^4u''''(x) + \frac{1}{120}32h^5u'''''(x) + ... \\ u(x+h) &= u(x) + hu'(x) + \frac{1}{2}h^2u''(x) + \frac{1}{6}h^3u'''(x) + \frac{1}{24}h^4u''''(x) + \frac{1}{120}h^5u'''''(x) + ... \\ u(x-h) &= u(x) - hu'(x) + \frac{1}{2}h^2u''(x) - \frac{1}{6}h^3u'''(x) + \frac{1}{24}h^4u''''(x) - \frac{1}{120}h^5u'''''(x) + ... \\ u(x-2h) &= u(x) - 2hu'(x) + \frac{1}{2}4h^2u''(x) - \frac{1}{6}8h^3u'''(x) + \frac{1}{24}16h^4u''''(x) - \frac{1}{120}32h^5u'''''(x) + ... \end{aligned}$$

A straightforward calculation implies $\frac{-u(x+2h)+8u(x+h)-8u(x-h)+u(x-2h)}{12h} = u'(x) - \frac{1}{30}h^4u'''''(x) + ...$ Thus, $b = -1/30$.

1.2.18. The general solution to $\frac{d^2u}{dx^2} = x$ is $u = \frac{1}{6}x^3 + Bx + C$. The boundary conditions $u(0) = u(1) = 0$ imply $C = 0, B = -\frac{1}{6}$,i.e. $u(x) = \frac{x^3-x}{6}$.

Hence $u(0.2) = -0.032, u(0.4) = -0.056, u(0.6) = -0.064, u(0.8) = -0.048$. The finite difference equation is: $-\frac{1}{h^2}K_4\begin{bmatrix} u_1 \\ \vdots \\ u_4 \end{bmatrix} = \begin{bmatrix} 0.2 \\ \vdots \\ 0.8 \end{bmatrix}$.

Hence $\begin{bmatrix} u_1 \\ \vdots \\ u_4 \end{bmatrix} = \frac{-1}{125}K_4^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{-1}{125}\cdot\frac{1}{5}\begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -.032 \\ -.056 \\ -.064 \\ -.048 \end{bmatrix}$,
which match the values for the actual solution exactly.