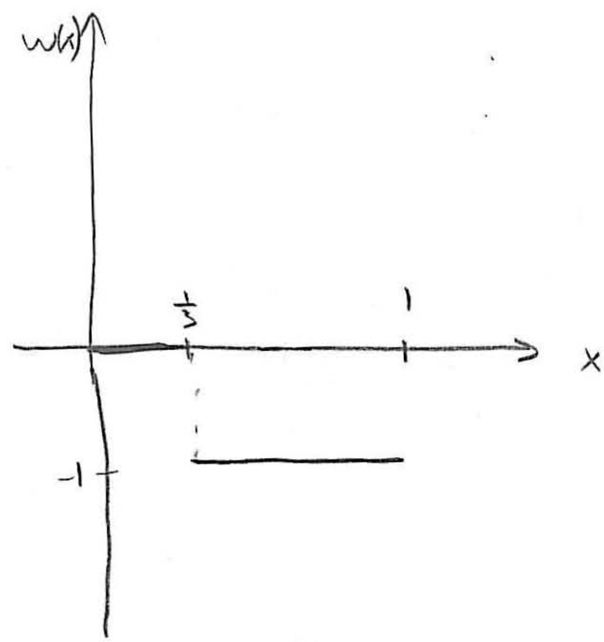
18.085 Quiz 2 2010

1 a) $w(x) = -\int_0^x \delta\left(x-\frac{1}{3}\right)dx = \begin{cases} C & x<\frac{1}{3} \\ -1+C & x>\frac{1}{3} \end{cases}$

$w(0)=0 \Rightarrow C=0$

$$\boxed{w(x) = \begin{cases} 0 & x<\frac{1}{3} \\ -1 & x>\frac{1}{3} \end{cases}}$$



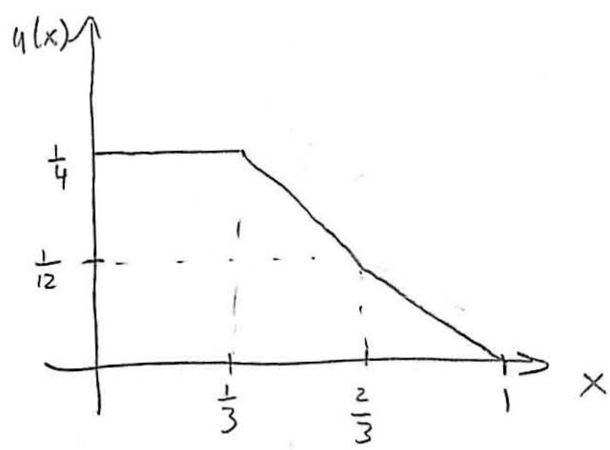
b) $u(x) = \int_0^x 0\,dx = C \qquad x<\frac{1}{3}$

$C+\int_{\frac{1}{3}}^x -\frac{1}{2}dx = C-\frac{1}{2}\left(x-\frac{1}{3}\right) = C+\frac{1}{6}-\frac{1}{2}x \qquad \frac{1}{3}<x\le\frac{2}{3}$

$C-\frac{1}{6}+\int_{\frac{2}{3}}^x -\frac{1}{4}dx = C-\frac{1}{6}-\frac{1}{4}\left(x-\frac{2}{3}\right) = C-\frac{1}{4}x \qquad \frac{2}{3}\le x<1$

$u(1) = C-\frac{1}{4} = 0 \Rightarrow C=\frac{1}{4}$

$$\boxed{u(x) = \begin{cases} \frac{1}{4} & x<\frac{1}{3} \\ \frac{5}{12}-\frac{1}{2}x & \frac{1}{3}<x\le\frac{2}{3} \\ \frac{1}{4}-\frac{1}{4}x & \frac{2}{3}\le x<1 \end{cases}}$$



2. Keep the same equation $-(cu')' = \delta(x - 1/3)$ as Problem 1 with jump in $c(x)$.

(a) What is the *weak form* of this problem? (Tell me the equation that holds for every test function $v(x)$.) Also tell me what boundary conditions are imposed on $u(x)$ and $v(x)$.

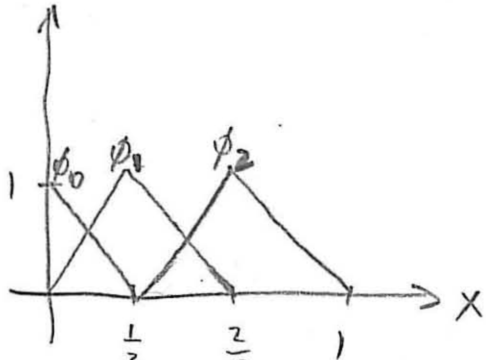
(b) Draw the graphs of three piecewise linear finite elements $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$ with step size $h = 1/3$, satisfying the appropriate boundary conditions on trial functions for this problem. Describe the graph of $U_0\,\phi_0(x) + U_1\,\phi_1(x) + U_2\,\phi_2(x)$.

(c) These ϕ_0, ϕ_1, ϕ_2 will be trial functions in $U(x) = U_0\,\phi_0(x) + U_1\,\phi_1(x) + U_2\,\phi_2(x)$. The same ϕ_0, ϕ_1, ϕ_2 will be the test functions V_0, V_1, V_2. From the weak form in part (a), with jump from $c = 2$ to $c = 4$ at $x = 2/3$, find the matrix K and right side F in the finite element equation $KU = F$. (*Not necessary* to solve for $U = (U_0, U_1, U_2)$.)

a) $-(cu')' = \delta(x - \frac{1}{3}) \Rightarrow \boxed{\int_0^1 cu'v'\,dx = \int_0^1 \delta(x-\tfrac{1}{3})\,v\,dx}$

for every test function $v(x)$ such that $u(1) = v(1) = 0$

b)



The graph of $U_0\phi_0(x) + U_1\phi_1(x) + U_2\phi_2(x)$ is the FEM approximation to $u(x)$ such that at each meshpoint $x = 0, \frac{1}{3}, \frac{2}{3}$ the graph equals U_0, U_1, and U_2, respectively and the graph is linear in between. In short, $U(x)$ is a piecewise linear function.

next → pg.

c) $K_{11} = \int_0^{\frac{1}{3}} c\,\phi_0'\phi_0'\,dx = \int_0^{\frac{1}{3}} 2\left(\frac{1}{h^2}\right) dx = \frac{2}{h} = 6$

$K_{12} = K_{21} = \int_0^{\frac{1}{3}} c\,\phi_0'\phi_1'\,dx = \int_0^{\frac{1}{3}} -2\left(\frac{1}{h^2}\right) dx = -\frac{2}{h} = -6$

$K_{22} = 2\left(\frac{2}{h}\right) = 12$

$K_{23} = K_{32} = -2\left(\frac{1}{h}\right) = -6$

$K_{33} = \int_{\frac{1}{3}}^{\frac{2}{3}} c\,\phi_2'\phi_2'\,dx + \int_{\frac{2}{3}}^{1} c\,\phi_2'\phi_2'\,dx = 2\left(\frac{1}{h}\right) + 4\left(\frac{1}{h}\right) = 18$

$K_{13} = K_{31} = 0$

$$\boxed{K = \begin{bmatrix} 6 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 18 \end{bmatrix}}$$

$F_1 = \int_0^1 \phi_0\,\delta(x-\tfrac{1}{3})\,dx = 0$

$F_2 = \int_0^1 \phi_1\,\delta(x-\tfrac{1}{3})\,dx = 1$

$F_3 = 0$

$$\boxed{F = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

3. (a) Find the gradient $v = \text{grad}\, u$ of $u(x,y) = x + x^2 - y^2$. Then set $w(x,y) = v(x,y)$. Then compute the divergence div $w(x,y)$. Then find a stream function $s(x,y)$ that solves $\partial s/\partial y = w_1(x,y)$ and $-\partial s/\partial x = w_2(x,y)$. Verify that u and s solve the Cauchy-Riemann equations. If $u + is$ is a function $F(x+iy)$, find F.

(b) (Connected to part (a)) Suppose Laplace's equation $-u_{xx} - u_{yy} = 0$ holds inside the unit circle $x^2 + y^2 = 1$, and on the boundary $u = u_0(\theta) = \cos\theta + \cos 2\theta$. What is the solution $u(r,\theta)$ at all points inside? What is that same solution written as $u(x,y)$? Find the flux $\int w \cdot n\, ds$ out of the circle, either directly (the vector $n = (x,y)$ points radially out) or by the Divergence Theorem.

a) $\vec{v} = \vec{w} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \boxed{\begin{bmatrix} 1+2x \\ -2y \end{bmatrix}}$

$\text{div}\, \vec{w}(x,y) = \frac{\partial \vec{w}}{\partial x} + \frac{\partial \vec{w}}{\partial y} = 2 - 2 = \boxed{0}$

$\frac{\partial s}{\partial y} = w_1 = 1 + 2x$

$\frac{\partial s}{\partial x} = -w_2 = 2y$

$\Rightarrow \boxed{s(x,y) = y + 2xy}$

$\frac{\partial u}{\partial x} = 1 + 2x = \frac{\partial s}{\partial y}$ ✓

$\frac{\partial u}{\partial y} = -2y = -\frac{\partial s}{\partial x}$ ✓

$u + is = x^2 + x + iy + 2xyi - y^2$

$F = (x+iy)^2 + (x+iy) = \boxed{(x+iy)(x+iy+1)}$

next pg.

b) $u(r,\theta) = \boxed{r\cos\theta + r^2\cos 2\theta}$

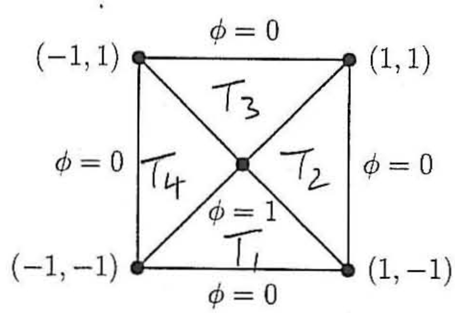
$$u(r,\theta) = r\cos\theta + r^2(2\cos^2\theta - 1) = r\cos\theta + 2(r\cos\theta)^2 - r^2$$

$$u(x,y) = x + 2x^2 - x^2 - y^2 = \boxed{x + x^2 - y^2}$$

$$\vec{w} = \begin{bmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} 1+2x \\ -2y \end{bmatrix}$$

$$\iint_{\text{circle}} \operatorname{div}\vec{w}\, dx\,dy = \boxed{0} = \text{Flux out of circle}$$

4. (a) Suppose $\phi(x, y)$ is a piecewise linear function with $\phi(0,0) = 1$ at the center of this square of side 2, and $\phi = 0$ around the boundary. Find the formula $a + bx + cy$ for $\phi(x, y)$ in each of the four parts of the square.



(b) Find the weak form of Poisson's equation with $u = 0$ along that square:

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \delta_{1/2,0} = \text{point load at } x = \frac{1}{2},\ y = 0.$$

Write this weak form neatly (symmetrically in u and v on the left side, explicitly on the right side).

(c) Use only that *one trial function* $\phi(x, y)$ (it is also the test function) from part (a). Substitute $U_1\,\phi$ for u in the weak form, and substitute ϕ for v. Integrate over the four parts of the square to find the equation for the number U_1 (here $KU = F$ is only one equation). Then $U(x, y) = U_1\,\phi(x, y)$ is the finite element approximation to the exact solution $u(x, y)$.

a) The four parts are labeled in the diagram above.

$T_1 : 1 + y$
$T_2 : 1 - x$
$T_3 : 1 - y$
$T_4 : 1 + x$

next pg.

b) $-u_{xx} - u_{yy} = \delta_{\frac{1}{2},0}$ $\Rightarrow$ $\boxed{\begin{array}{l} \iint_{\text{square}} u_x v_x + u_y v_y \, dx\,dy = \iint_{\text{square}} \delta_{\frac{1}{2},0} \, v dx dy \\ \qquad = v(\tfrac{1}{2}, 0) \\ \text{for } v=0 \text{ along boundary of square} \end{array}}$

c) Parts of the square are labeled the same as in part (a).

Over T_1: $\iint_{T_1} U_1 \phi_x \phi_x + U_1 \phi_y \phi_y \, dx dy = \iint_{T_1} \delta_{\frac{1}{2},0} \, \phi \, dx dy$

$+ \qquad\qquad +$

Over T_2: $\iint_{T_2} U_1 \phi_x \phi_x + U_1 \phi_y \phi_y \, dx dy = \iint_{T_2} \delta_{\frac{1}{2},0} \, \phi \, dx dy$

$+ \qquad\qquad +$

Over T_3: $\iint_{T_3} U_1 \phi_x \phi_x + U_1 \phi_y \phi_y \, dx dy = \iint_{T_3} \delta_{\frac{1}{2},0} \, \phi \, dx dy$

$+ \qquad\qquad +$

Over T_4: $\iint_{T_4} U_1 \phi_x \phi_x + U_1 \phi_y \phi_y \, dx dy = \iint_{T_4} \delta_{\frac{1}{2},0} \, \phi \, dx dy$

$\Downarrow \qquad\qquad \Downarrow$

$$U_1(1)(1) + U_1(1)(1) + U_1(1)(1) + U_1(1)(1) = 0 + \tfrac{1}{2} + 0 + 0$$

$$4U_1 = \tfrac{1}{2}$$

$$\boxed{\begin{array}{c} U_1 = \tfrac{1}{8} \\ U(x,y) = \tfrac{1}{8}\phi(x,y) \end{array}}$$