18.085 Exam 1 October 15, 2010 Professor Strang

Your PRINTED name is: ______________________________ 1. 15

Your class number (1 to 155) is ______________________________ 2. 25

3. 30

4. 30

100

1. Let us agree to use the *energy-based definition* of a positive semidefinite (symmetric) matrix: K is positive semidefinite *means that* $x^T Kx \geq 0$ for all vectors x.

(a) If $K = A^T A$ for some matrix A, show that this K is positive semidefinite. (K might actually be positive definite, just show semidefinite.)

OR

For any vector x, $x^T K x = x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 \geq 0$

$\Rightarrow$ K is positive semidefinite.

(b) If K is positive semidefinite and $Kx = \lambda x$ for some x, show that $\lambda \geq 0$.

For a non-zero vector x, $\|x\|^2 > 0$ $\Rightarrow$ using the pos. semidefiniteness of K, $0 \leq \frac{x^T K x}{\|x\|^2} = \frac{\lambda x^T x}{\|x\|^2} = \lambda$.

(c) For which numbers c is this matrix positive semidefinite? Use the determinant tests.

$$K = \begin{bmatrix} 2 & -1 & 0 \\ -1 & c & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

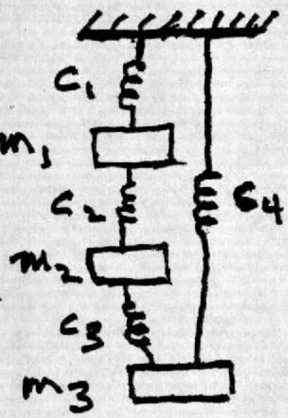
K will be positive ~~[illegible]~~ semidefinite once

1) $2 \geq 0$ ✓

2) $\det \begin{pmatrix} 2 & -1 \\ -1 & c \end{pmatrix} = 2c - 1 \geq 0$

3) $\det K = 2(2c-1) + (-2) = 4c - 4 \geq 0$.

(1), (2) and (3) are satisfied whenever $c \geq 1$.

2. 

A fixed-free line of springs has one extra spring from the fixed top to the free mass m_3 at the bottom (see picture).

(a) What is the matrix A that gives the stretching $e = Au$ in the springs from the displacements u_1, u_2, u_3 of the masses?

A is the $\underset{\#\text{springs}}{4} \times \underset{\#\text{masses}}{3}$ matrix $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(b) What is the matrix $K = A^T CA$ in the final equation $Ku = f$ (f = external forces on the masses)?

Either we compute directly or we add up the element matrices to get $K = \begin{pmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{pmatrix}$

(c) What do you know about the eigenvalues of K? If $Kx = \lambda x$, tell me one solution (not $u = 0$) of the vector equation $\dfrac{d^2u}{dt^2} + Ku = 0$. $(*)$

We are looking for a solution to Newton's eqn for the given system with $m_1 = m_2 = m_3 = 1$.

K is a positive definite matrix, therefore its eigenvalues are positive (there are several ways to show K is pos. def., e.g. take a non-zero 3×1 vector $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and note that $Ax \neq 0$ since the columns of A are linearly independent $\Rightarrow \sqrt{C}\,Ax \neq 0 \Rightarrow x^T K x = (\sqrt{C}\,Ax)^T (\sqrt{C}\,Ax) > 0$)

The general solution to $(*)$ is a linear combination of the normal modes of the system: one of them is

$$\boxed{u(t) = \cos(\sqrt{\lambda}\,t - \varphi)\, x}$$

3. Suppose you want the least squares fit to FIVE points in the plane by a parabola $C + Dt + Et^2$.

(a) If a parabola does have the required values $b_1, b_2, \ldots, b_5$ when $t = 0, 1, 2, 3, 4$, what system of equations has a solution?

$$A \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

<u>Write the complete matrix A.</u>

$$\begin{array}{cc} & \begin{array}{ccc} 1 & t & t^2 \\ \downarrow & \downarrow & \downarrow \end{array} \\ A = & \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix} \end{array}$$

What equation (use letters like A, not all its entries) gives the least squares solution $\hat{u} = (\widehat{C}, \widehat{D}, \widehat{E})$?

The normal eqn: $A^T A \hat{u} = A^T b \quad (*)$

(b) Draw a picture to show this problem in 5-dimensional space! Include the vector b and the columns of A. Show the projection p of b onto the "subspace" that contains those columns of A.

(see below)

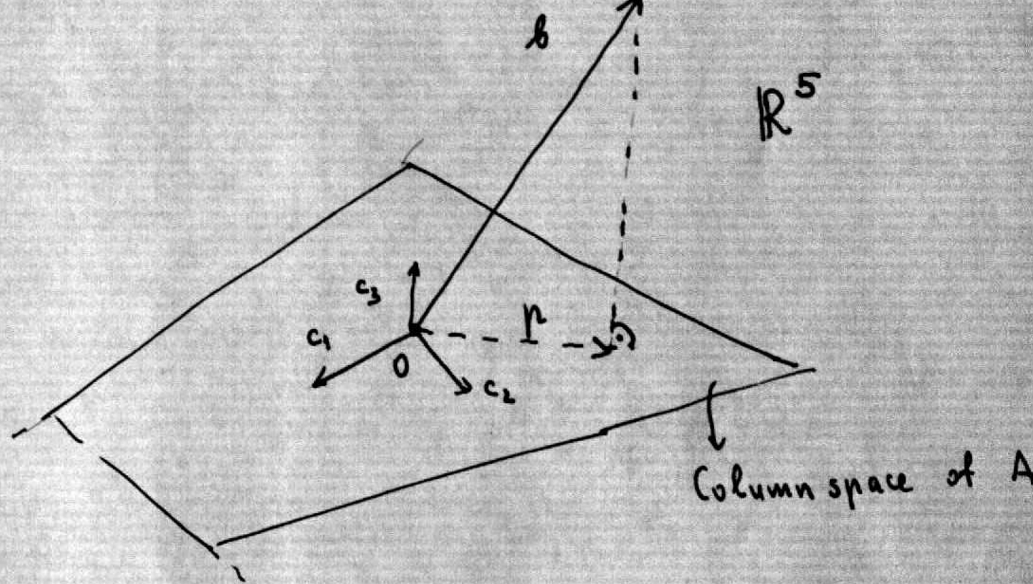
(c) *How is the vector p (the projection) related to the matrix A and the solution $\hat{u}$?*

$p = A \hat{u}$

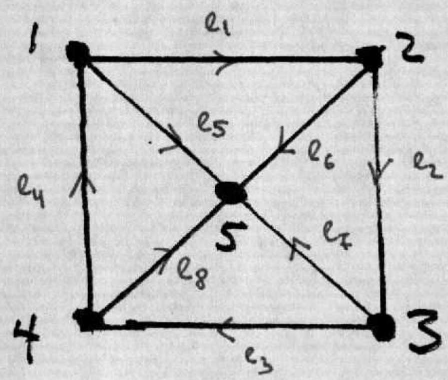
(d) For which measurements b will the best parabola be the zero parabola, $\widehat{C} = \widehat{D} = \widehat{E} = 0$? Show how b is related to the columns of A in this case.

We want to find those b, for which the solution to $(*)$ is $\hat{u} = 0 \Rightarrow A^T b = 0 \Rightarrow b$ is <u>orthogonal</u> to the columns of A. Geometrically this makes perfect sense, because the projection of b onto the plane spanned by the columns of A needs to be 0.

b)



4. This square network plus center point has 5 nodes and 8 edges. Its incidence matrix is A. The constants are $c_i = 1$ on all edges, so $C = I$.



(a) What is the matrix $K = A^T A$? (You can number edges and put in arrows as you want, K is only involved with nodes.)

$$K = A^T A = \underset{\text{degree m.}}{D} - \underset{\text{adjacency m.}}{W} = \begin{pmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}$$

(b) Is K invertible or singular? Explain how you know you are right.

We see that $K \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0 \Rightarrow K$ is singular.

(c) How many independent solutions are there to Kirchhoff's Current Law $A^T w = 0$? Describe one set of currents w that solves this equation.

There are 4 independent loop flows $\Rightarrow$ 4 indep. sol'ns to $A^T w = 0$

With the chosen labelling of edges above, one of them is $w = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

(d) Suppose current sources f and voltage sources b are zero. Fix the center node voltage at $u = 0$ (grounded) AND fix the top left node at $u = 1$. What equations would you solve for the three unknown $u's$? (Do not solve.) Is your matrix invertible?

The fundamental eqn $\underbrace{A^T C A}_{K} u = A^T C b - f$ (pp. 143 in the book)

becomes

$$K \begin{pmatrix} 1 \\ u_2 \\ u_3 \\ u_4 \\ 0 \end{pmatrix} = -f \quad \text{where } f_2 = f_3 = f_4 = 0$$

(the non-zero current sources are only 1 and 5)

$$\Rightarrow 1 \begin{pmatrix} 3 \\ -1 \\ 0 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & -1 \\ 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = - \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \\ * \end{pmatrix} \Rightarrow$$

$$\Rightarrow \underbrace{\begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}}_{M} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

We compute $\det M = 3.8 - 3 = 21$

$\Rightarrow M$ is invertible.