SOLUTIONS TO THE SUGGESTED PROBLEMS FOR EXAM 3

4.3.15 Since the inverse of $F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$ is $F^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{bmatrix}$, one easily computes

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix} = F^{-1}\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix} F = \begin{bmatrix} 1 & & & \\ & i & & \\ & & -1 & \\ & & & -i \end{bmatrix}$$

4.4.1 By the definition of convolution, we have $(12) * (15) = (1\cdot 1, 1\cdot 5 + 2\cdot 1, 2\cdot 5) = (1, 7, 10)$. Thus, $t = 10$.

4.4.2 We have

$$(Fc).*(Fd) = \begin{bmatrix} c_0 + c_1 \\ c_0 - c_1 \end{bmatrix} .* \begin{bmatrix} d_0 + d_1 \\ d_0 - d_1 \end{bmatrix} = \begin{bmatrix} (c_0+c_1)(d_0+d_1) \\ (c_0-c_1)(d_0-d_1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_0d_0 + c_1d_1 \\ c_0d_1 + c_1d_0 \end{bmatrix} = F(c*d).$$

4.4.6 The identity vector is $\delta_N = (1, 0, ..., 0)$ with $N-1$ zeros.

4.4.8 (a) f corresponds to w^3 where $w^N = w^6 = 1$. Hence $f*f$ corresponds to $w^3 \cdot w^3 = 1$, i.e. $f*f = (1,0,0,0,0,0)$.

(b) The discrete transform of f is

$$c = F_6^{-1} f = \text{4th column of } F_6^{-1} = \frac{1}{6}\begin{bmatrix} 1 \\ w^{-3} \\ w^{-6} \\ w^{-9} \\ w^{-12} \\ w^{-15} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

(c) We have

$$f*f = 6F_6(c.*c) = 6\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 & w^4 & w^5 \\ 1 & w^2 & w^4 & w^6 & w^8 & w^{10} \\ 1 & w^3 & w^6 & w^9 & w^{12} & w^{15} \\ 1 & w^4 & w^8 & w^{12} & w^{16} & w^{20} \\ 1 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} \end{bmatrix} \frac{1}{36}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4.5.1 We have $\hat{g}(k) = \int_{-\infty}^{\infty} g(x)e^{-ikx}dx = \int_{-\infty}^{0} -e^{(a-ik)x}dx + \int_0^\infty e^{(-a-ik)x}dx = -\frac{1}{a-ik} + \frac{1}{a+ik} = \frac{-2ik}{a^2+k^2}$. The decay rate of $\hat{g}(k)$ is $\frac{1}{k}$. There is a discontinuity in $g(x)$.

4.5.2 (a) $\hat{f}(k) = \int_0^L e^{-ikx}dx = \frac{1-e^{-iLx}}{ik}$

(b) Setting a=0 in **4.5.1** we find $\hat{f}(k) = \frac{-2i}{k}$

(c) Note that $f(x) = \int_0^1 e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(k)e^{ikx}dk$, where

$$g(k) = \begin{cases} 2\pi & \text{for} \quad 0 \le k \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

We immediately recognize this expression as the Inverse Fourier Transform. Thus, $\hat{f} = g$.

(d) $\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx}\sin x dx = \frac{e^{-ikx}}{k^2-1}(\cos x + ik\sin x)|_0^{4\pi} = \frac{e^{-4\pi ik}-1}{k^2-1}$

4.5.10 The reason is because $df/dx = -af + \delta(x)$, and not just $df/dx = -af$. Hence, the transform of df/dx could be computed as $-a\hat{f}(k) + 1 = \frac{-a}{a+ik} + 1 = \frac{ik}{a+ik}$, which agrees with $ik\hat{f}(k)$.

4.5.11 Taking Fourier transforms of both sides we obtain:

$$ik\hat{u}(k) + a\hat{u}(k) = e^{-ikd} \Rightarrow \hat{u}(k) = \frac{e^{-ikd}}{a + ik}$$

If $v(x) = u(x+d)$ then $\hat{v}(k) = e^{ikd}\hat{u}(k) = \frac{1}{a+ik}$. Hence we find $v(x) = \left\{ \begin{array}{c} e^{-ax} \ x \geq 0 \\ 0 \ x < 0 \end{array} \right\}$and $u(x) = \left\{ \begin{array}{c} e^{-a(x-d)} \ x \geq d \\ 0 \ x < d \end{array} \right\}$.

4.5.12 The Fourier transform rules give:

$$\frac{\hat{u}(k)}{ik} - ik\hat{u}(k) = 1 \Rightarrow \hat{u}(k) = \frac{ik}{1 + k^2}$$

Hence, **4.5.1** tells us that u is an odd two-sided pulse: $u = \left\{ \begin{array}{c} -e^{-x}/2 \ x > 0 \\ e^{x}/2 \ x < 0 \end{array} \right\}$