4.1: 1, 3, 5, 7, 13

1) Find the Fourier series on $-\pi \le x \le \pi$ for ...

(a) $f(x) = \sin^3 x$, an odd function

$$\begin{aligned}
\sin(3x) &= \sin(x)\cos(2x) + \cos(x)\sin(2x) \\
&= \sin(x)\left[\cos^2(x) - \sin^2(x)\right] + 2\cos^2(x)\sin(x) \\
&= \sin(x) - 2\sin^3(x) + 2\sin(x) - 2\sin^3(x) \\
\sin(3x) &= 3\sin(x) - 4\sin^3(x)
\end{aligned}$$

$$\boxed{\sin^3(x) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)}$$

(b) $f(x) = |\sin x|$, an even function

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin(x)| dx = \frac{1}{\pi}\int_0^{\pi} \sin(x) dx = \frac{1}{\pi}\left[-\cos(\pi) + \cos(0)\right] = \frac{2}{\pi}$$

First, the identity $2\sin(x)\cos(kx) = \sin((k+1)x) - \sin((k-1)x)$ for $k = 1, 2, ...$ is proven, which will be useful later.

$$\sin((k+1)x) - \sin((k-1)x) = \cancel{\sin(kx)\cos(x)} + \cos(kx)\sin(x) - \cancel{\sin(kx)\cos(x)} + \cos(kx)\sin(x) = 2\sin(x)\cos(kx)$$

For $k = 1, 2, ...$:

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(x)|\cos(kx) dx = \frac{2}{\pi}\int_0^{\pi}\sin(x)\cos(kx)dx = \frac{1}{\pi}\int_0^{\pi}\left[\sin((k+1)x) - \sin((k-1)x)\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{k+1}\cos((k+1)x) + \frac{1}{k-1}\cos((k-1)x)\right]_0^{\pi} \\
&= \frac{1}{\pi}\left[-\frac{1}{k+1}\cos((k+1)\pi) + \frac{1}{k-1}\cos((k-1)\pi) + \frac{1}{k+1}\cos(0) - \frac{1}{k-1}\cos(0)\right] \\
&= \begin{cases} 0, & \text{if } k \text{ is odd} \\ \frac{2}{\pi}\left[\frac{1}{k+1} - \frac{1}{k-1}\right], & \text{if } k \text{ is even} \end{cases} = \begin{cases} 0, & \text{if } k \text{ is odd} \\ -\frac{4}{\pi}\left[\frac{1}{k^2-1}\right], & \text{if } k \text{ is even} \end{cases}
\end{aligned}$$

$$\boxed{|\sin(x)| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\cos((2j)x)}{(2j)^2 - 1}}$$

(c) $f(x) = x$, an odd function which can be treated as periodic over $-\pi$ to π

$$\begin{aligned}
b_k &= \frac{2}{\pi}\int_0^{\pi} x\sin(kx)dx = \frac{2}{\pi}\int_0^{\pi} x\frac{d}{dx}\left[-\frac{1}{k}\cos(kx)\right]dx = \frac{2}{k\pi}\int_0^{\pi}\frac{d}{dx}[x]\cos(kx)dx + \frac{2}{\pi}\left[-\frac{x}{k}\cos(kx)\right]_0^{\pi} \\
&= \frac{2}{k^2\pi}\left[\sin(k\pi) - \sin(0)\right] - \frac{2}{\cancel{\pi}}\left[\frac{\cancel{\pi}}{k}\cos(k\pi) + 0\right] = -\frac{2(-1)^k}{k}
\end{aligned}$$

$$\boxed{x = 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\sin(kx)}{k} \quad \text{for } -\pi < x < \pi}$$

(d) $f(x) = e^x$, using the complex form of the series

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-ikx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(1-ik)} dx = \frac{1}{2\pi}\left[\frac{1}{1-ik}e^{x(1-ik)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(1-ik)}\left[e^{\pi}(\cos(k\pi) - \cancel{i\sin(k\pi)}) - e^{-\pi}(\cos(k\pi) + \cancel{i\sin(k\pi)})\right] = \frac{(-1)^k(e^{\pi} - e^{-\pi})}{2\pi(1-ik)}$$

$$\boxed{e^x = \frac{e^{\pi} - e^{-\pi}}{2\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^k e^{ikx}}{ik-1}}$$

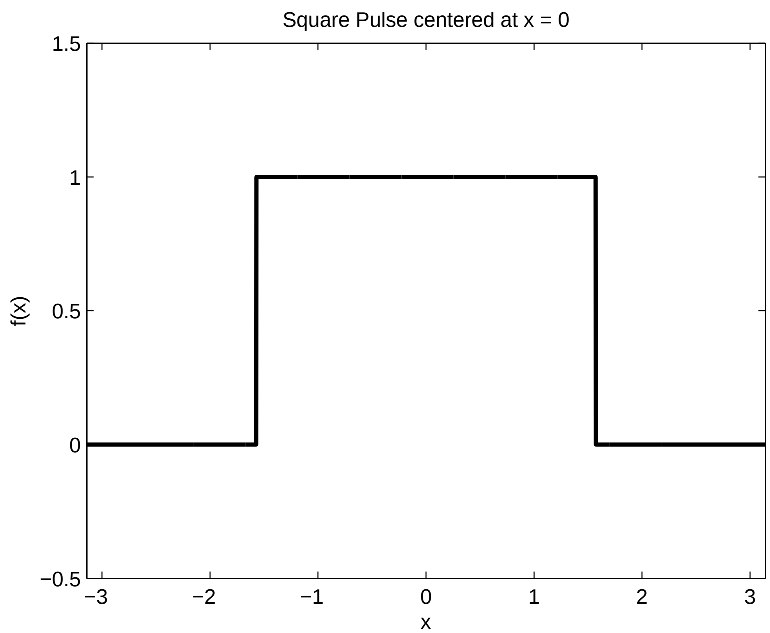
A function $F(x)$ can be split into even, $F_{\text{even}}(x)$, and odd, $F_{\text{odd}}(x)$, parts.

$$F(x) = F_{\text{even}}(x) + F_{\text{odd}}(x) \qquad F_{\text{even}}(x) = \frac{F(x) + F(-x)}{2} \qquad F_{\text{odd}}(x) = \frac{F(x) - F(-x)}{2}$$

The even an odd parts of e^x are $\cosh(x)$ and $\sinh(x)$, respectively.
The even and odd parts of e^{ix} are $\cos(x)$ and $i\sin(x)$, respectively.

3) A periodic square pulse centered at $x = 0$ is given by the following equation for $-\pi < x < \pi$:

$$f(x) = \begin{cases} 1, & \text{for } |x| < \frac{\pi}{2} \\ 0, & \text{for } \frac{\pi}{2} < |x| < \pi \end{cases}$$



We would like to find the Fourier coefficients a_k, b_k for this square pulse. Because the pulse is an even function, $\boxed{b_k = 0}$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx = \frac{1}{2\pi}\pi = \boxed{a_0 = \frac{1}{2}}$$

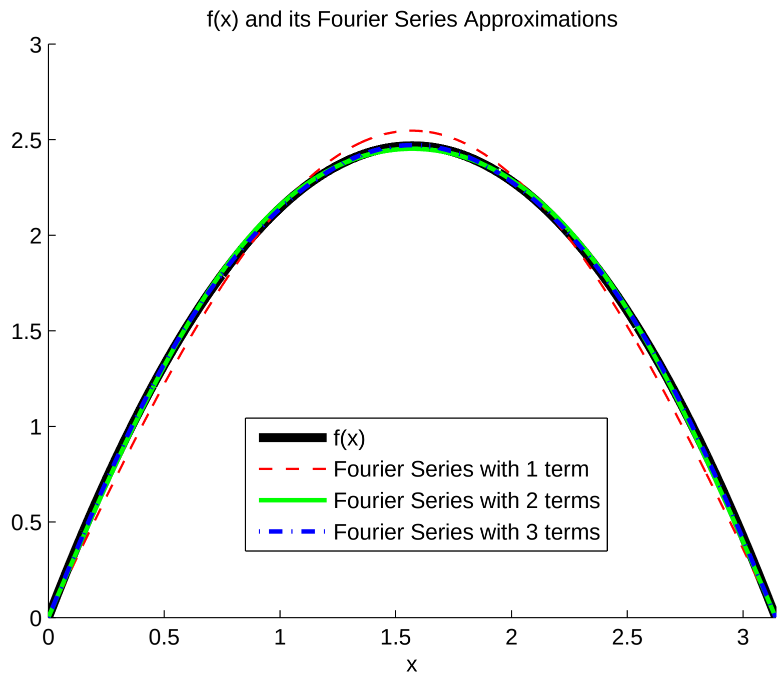
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(kx)dx = \frac{1}{k\pi}\left[\sin\left(\frac{k\pi}{2}\right) - \sin\left(-\frac{k\pi}{2}\right)\right] = \boxed{a_k = \frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right)}$$

5) The function $f(x) = x(\pi - x)$ is written in terms of its Fourier series below.

$$f(x) = x(\pi - x) = \frac{8}{\pi}\left(\frac{\sin(x)}{1} + \frac{\sin(3x)}{27} + \frac{\sin(5x)}{125} + ...\right),\ 0 < x < \pi$$

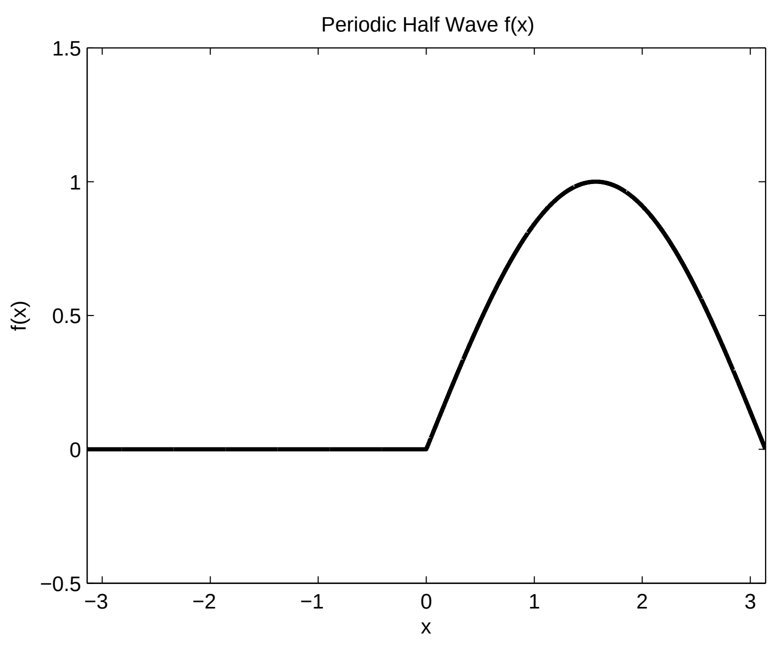
The decay rate is $1/k^3$, and this can be seen by first taking the second derivative of the function $f(x)$. $f''(x) = -2$ for $0 < x < \pi$, and because $f(x)$ has to be an odd function over $-\pi < x < \pi$, $f''(x) = 2$ for $-\pi < x < 0$. Clearly, the second derivative is a step function with an equation given by $f''(x) = 2 - 4S(x)$ for $-\pi < x < \pi$. Since the step function has a decay rate of $1/k$, and each time a function is integrated the decay rate becomes $^1/_k$ times faster, the function $f(x)$ must have a decay rate of $1/k^3$.

The function is plotted below along with its first three partial sums.



7)

$$f(x) = \begin{cases} \sin(x), & \text{for } 0 < x < \pi \\ 0, & \text{for } -\pi < x < 0 \end{cases}$$



The Fourier coefficient of $f(x)$ are:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{0}^{\pi} \sin(x)dx = \frac{1}{2\pi}\left[-\cos(\pi)+\cos(0)\right] = \frac{1}{\pi}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\cos(kx)dx = \begin{cases} 0, & \text{if } k \text{ is odd} \\ -\frac{2}{\pi}\left[\frac{1}{k^2-1}\right] & \text{if } k \text{ is even} \end{cases} \qquad \text{(See Problem 1b)}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx = \frac{1}{\pi}\int_{0}^{\pi} \sin(x)\sin(kx)dx = \begin{cases} \frac{1}{2}, & \text{if } k = 1 \\ 0, & \text{if } k \neq 1 \end{cases}$$

$$\boxed{f(x) = \frac{1}{2}\sin(x) + \frac{1}{\pi} - \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{\cos((2j)x)}{(2j)^2-1}}$$

Comparing the above Fourier series to that in problem 1b, it can be seen that the above equation is simply $f(x) = \frac{1}{2}\sin(x) + \frac{1}{2}|\sin(x)|$.

13) If the centered square pulse in Example 7 has width $h = \pi$, we get the same function as the one in problem 3. Its formula is repeated below.

$$f(x) = \begin{cases} 1, & \text{for } |x| < \frac{\pi}{2} \\ 0, & \text{for } \frac{\pi}{2} < |x| < \pi \end{cases}$$

(a) The energy of the function can be calculated by direction integration.

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx = \boxed{\pi}$$

(b) The Fourier coefficients c_k of $f(x)$ are (for $k \neq 0$):

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-ikx}dx = \frac{1}{-2ik\pi}\left[\exp\left(-\frac{ik\pi}{2}\right) - \exp\left(\frac{ik\pi}{2}\right)\right] = \boxed{c_k = \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right)}$$

And for $k = 0$, $\boxed{c_0 = 1/2}$.

(c) Another way to calculate the energy of $f(x)$ is through the Fourier coefficients.

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi\sum_{k=-\infty}^{\infty}|c_k|^2 = 2\pi|c_0|^2 + 4\pi\sum_{k=1}^{\infty}|c_k|^2 = \frac{\pi}{2} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin^2(k\pi/2)}{k^2} = \frac{\pi}{2} + \frac{4}{\pi}\underbrace{\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2}}_{=\frac{\pi^2}{8}\ \text{(due to wolframalpha.com)}}$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\pi^2}{8}\right) = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

If $h = 2\pi$, the square pulse is always 1 over the entire interval of $-\pi < x < \pi$, therefore all the Fourier coefficients would be zero except for $c_0 = 1$, and $F(x)$ would be $F(x) = 1$.

4.2: 3

3) A 2π-periodic, odd 2D function $F(x,y)$ has a double sine series:

$$F(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(nx)\sin(my)$$

The coefficients b_{nm} can be found through the formula

$$\boxed{b_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(x,y)\sin(nx)\sin(ny)dxdy}$$

4.3: 2, 6, 10, 11

2) The jth row (where $j=0$ is the first row and $j=N-1$ is the last row) of the Fourier matrix F_N is a row vector

$$\begin{bmatrix} 1 & w^j & w^{2j} & \dots & w^{(N-1)j}\end{bmatrix} \tag{1}$$

where $w = \exp(i2\pi/N)$. The $(N-j)$th row vector of F_N is

$$\begin{bmatrix} 1 & w^{(N-j)} & w^{2(N-j)} & \dots & w^{(N-1)(N-j)}\end{bmatrix}$$

which can be also written as

$$\begin{bmatrix} 1 & \overline{w^j} & \overline{w^{2j}} & \dots & \overline{w^{(N-1)j}}\end{bmatrix} \tag{2}$$

by noting that the complex conjugate of w^{kj} is $\overline{w^{kj}} = w^{k(N-j)}$. The row vector 2 is simply the complex conjugate of the row vector 1, or alternatively, the $(N-j)$th row of F_N is identical to the jth row of the $\overline{F}_N$. The proof of $\overline{w^{kj}} = w^{k(N-j)}$ is given below:

$$\begin{aligned}\overline{w^{kj}} &= \left(\overline{w^k}\right)^j = \left(\overline{\exp\left(\frac{i2\pi k}{N}\right)}\right)^j = 1^k\left(\exp\left(\frac{-i2\pi k}{N}\right)\right)^j = \left(\exp\left(\frac{i2\pi N}{N}\right)\right)^k\left(\exp\left(\frac{-i2\pi k}{N}\right)\right)^j \\ &= \exp\left(\frac{i2\pi kN}{N}\right)\exp\left(-\frac{i2\pi kj}{N}\right) = \exp\left(\frac{i2\pi k(N-j)}{N}\right) = \left[\exp\left(\frac{i2\pi}{N}\right)\right]^{k(N-j)} = w^{k(N-j)}\end{aligned}$$

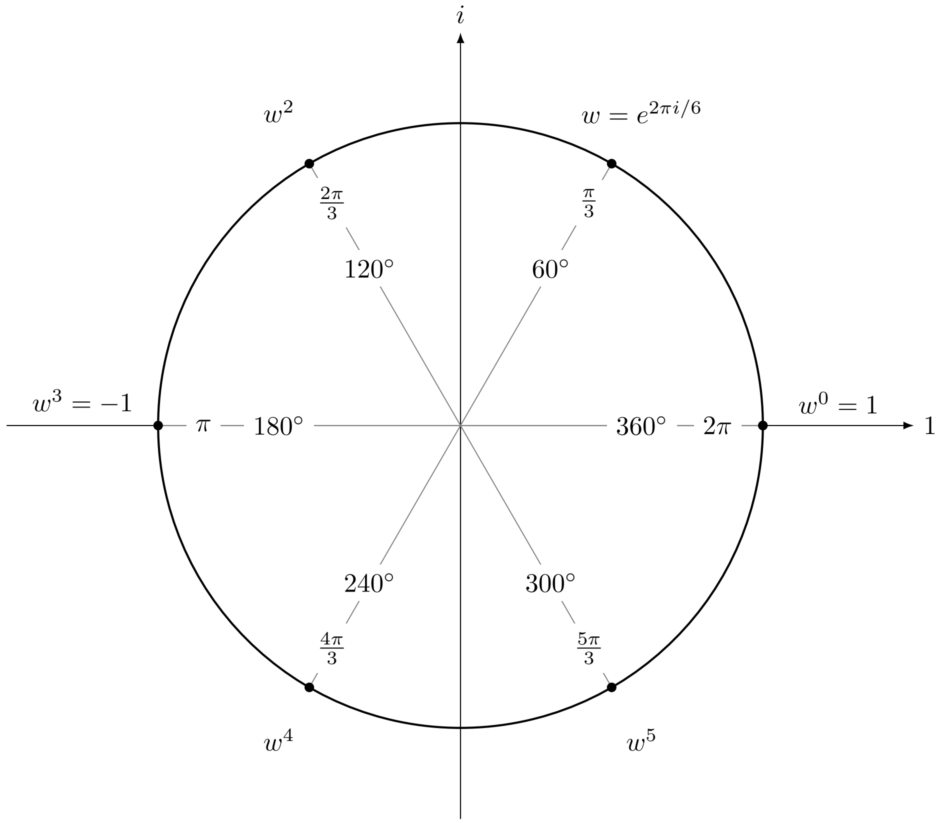
6) $w = \exp\left(\frac{i2\pi}{6}\right)$, and

$$\begin{aligned} F_6 &= \begin{bmatrix} I & D \\ I & -D\end{bmatrix}\begin{bmatrix} F_3 & \\ & F_3\end{bmatrix}\begin{bmatrix}\text{even}\\ \text{odd}\end{bmatrix} \\ &= \begin{bmatrix} 1&0&0&1&0&0\\ 0&1&0&0&w&0\\ 0&0&1&0&0&w^2\\ 1&0&0&-1&0&0\\ 0&1&0&0&-w&0\\ 0&0&1&0&0&-w^2\end{bmatrix}\begin{bmatrix} 1&1&1&0&0&0\\ 1&w^2&w^4&0&0&0\\ 1&w^4&w^2&0&0&0\\ 0&0&0&1&1&1\\ 0&0&0&1&w^2&w^4\\ 0&0&0&1&w^4&w^2\end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\end{bmatrix} \\ &= \begin{bmatrix} 1&1&1&1&1&1\\ 1&w^2&w^4&w&-1&-w^2\\ 1&w^4&w^2&w^2&1&w^4\\ 1&1&1&-1&-1&-1\\ 1&w^2&w^4&-w&1&w^2\\ 1&w^4&w^2&-w^2&-1&-w^4\end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\end{bmatrix} = \begin{bmatrix} 1&1&1&1&1&1\\ 1&w&w^2&-1&-w&-w^2\\ 1&w^2&-w&1&w^2&-w\\ 1&-1&1&-1&1&-1\\ 1&-w&w^2&1&-w&w^2\\ 1&-w^2&-w&-1&w^2&w\end{bmatrix}\end{aligned}$$

10) If $w = \exp\left(\frac{2\pi i}{64}\right)$ then w^2 and $\sqrt{w}$ are among the $\underline{32^{\text{th}}}$ and $\underline{128^{\text{th}}}$ roots of 1, respectively.

11)

(a) The sixth roots of 1 are: $1, \exp\left(i2\pi\frac{1}{6}\right), \exp\left(i2\pi\frac{2}{6}\right), \exp\left(i2\pi\frac{3}{6}\right), \exp\left(i2\pi\frac{4}{6}\right), \exp\left(i2\pi\frac{5}{6}\right)$. They are drawn on the unit circle below.



It can be shown that the sum of these six roots is zero:

$$\begin{aligned}\text{Sum of the roots} &= 1 + \exp\left(i2\pi\frac{1}{6}\right) + \exp\left(i2\pi\frac{2}{6}\right) + \exp\left(i2\pi\frac{3}{6}\right) + \exp\left(i2\pi\frac{4}{6}\right)\exp\left(i2\pi\frac{5}{6}\right)\\ &= 1 + \exp\left(\frac{i\pi}{3}\right) + \exp\left(\frac{i4\pi}{3}\right) - 1 - \exp\left(\frac{i4\pi}{3}\right) - \exp\left(\frac{i\pi}{3}\right)\\ &= 0\end{aligned}$$

(b) The three cube roots of 1 are: $1, \exp\left(i2\pi\frac{1}{3}\right), \exp\left(i2\pi\frac{2}{3}\right)$. These roots also add to zero:

$$\begin{aligned}\text{Sum of the roots} &= 1 + \exp\left(i2\pi\frac{1}{3}\right) + \exp\left(i2\pi\frac{2}{3}\right) = 1 + \exp\left(\frac{i2\pi}{3}\right) + \exp\left(-\frac{i2\pi}{3}\right)\\ &= 1 + 2\cos\left(\frac{2\pi}{3}\right) = 1 + 2\left(-\frac{1}{2}\right) = 0\end{aligned}$$

MATLAB Assignment

The square wave is given by the periodic function $f(x)$, listed below along with its Fourier sine series.

$$f(x) = \begin{cases} -1, & \text{for } -\pi < x < 0 \\ 1, & \text{for } 0 < x < \pi \end{cases} \qquad f(x) = \frac{4}{\pi}\left[\frac{\sin(x)}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} + \dots\right]$$

The partial sum of the Fourier series is plotted along with the square wave in Figure 1. The overshoot, undershoot, and width of the overshoot is tabulated for various partial sums in Table 1. The code used to generate these plots, and calculate information about the Gibbs phenomenon, is displayed in Listing 1.

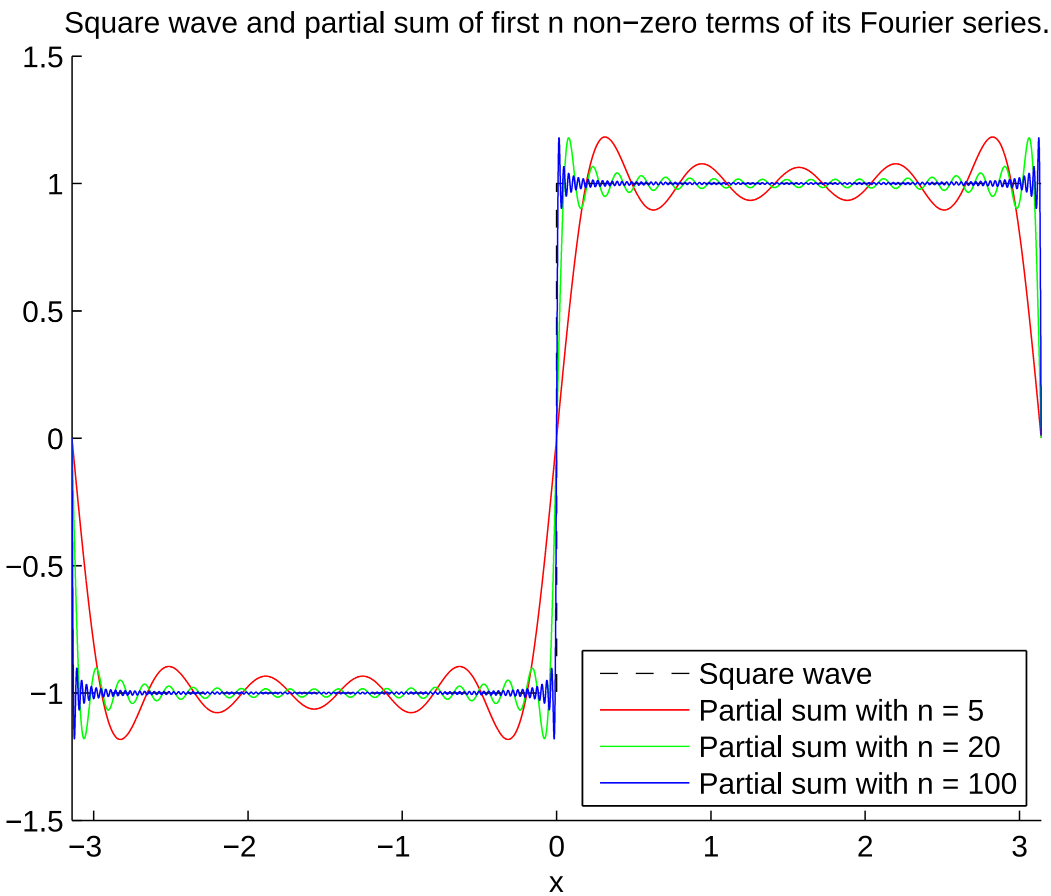


Figure 1: Plot of square wave and partial sum of Fourier series.

Table 1: Overshoot, undershoot, and width of overshoot.

n	5	10	20	50	100	200
Overshoot	0.18233	0.17981	0.17919	0.17901	0.17899	0.17895
Undershoot	0.10411	0.09886	0.097594	0.097243	0.097193	0.097181
Width	0.2958	0.1482	0.0742	0.0297	0.0148	0.0074

As seen in Figure 1 and Table 1, the overshoot and undershoot are not vanishing as the number of terms in the partial sum is increased. However, the width of the overshoot does get increasingly smaller as the number of terms, n, increases. This relationship is plotted in Figure 2, along with a curve fit of $1.4664/n$.

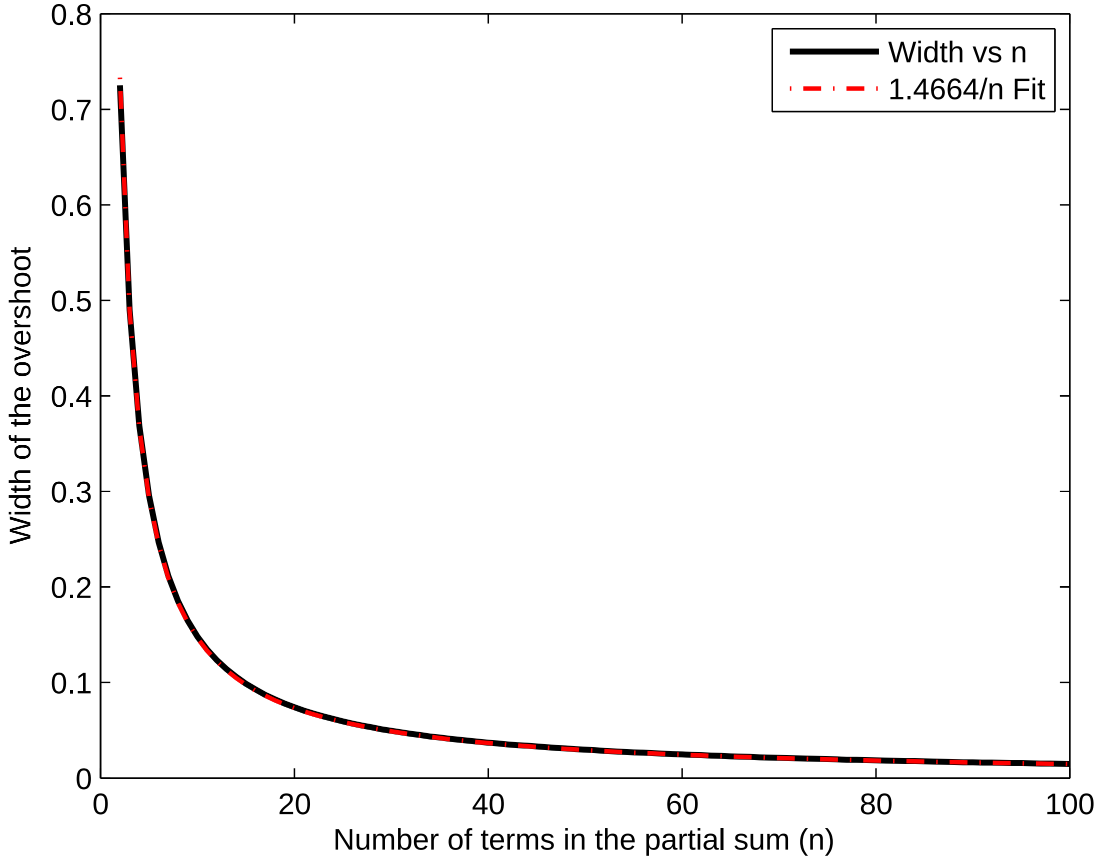


Figure 2: Relationship between the width of the overshoot and the number of terms in the partial sum of the Fourier series.

Listing 1: MATLAB code to plot Fourier series approximation of square wave and quantitatively describe the resulting Gibbs phenomenon.

```
% Domain is -pi < x < pi
x = [-pi:0.0001:pi]';

% Square wave
SW = ones(size(x));
SW(find(x<0)) = -1;

% Partial sum of Fourier series with first n non-zero terms
n = [5, 20, 100];
FS = zeros(length(x), length(n));
for i = 1:length(n)
    for j = 1:n(i)
        k = 2*j - 1;
        FS(:,i) = FS(:,i) + sin(k*x) / k;
    end
end
FS = FS * 4 / pi;

```

```
% Calculate overshoot, undershoot, and width of the overshoot.
OS = zeros(length(n), 1);
US = zeros(length(n), 1);
W = zeros(length(n), 1);
for i=1:length(n)
    indx = find(FS(:,i) > 0, 1); % Go to beginning of positive half.
    % Go to where it first crosses y = 1.
    indx = indx - 1 + find(FS(indx:end,i) >= 1, 1);
    width_start = indx; % Remember this location.
    % Go to the next local maximum (this is the overshoot).
    indx = indx - 1 + find(FS(indx+1:end,i) < FS(indx:end-1,i), 1);
    OS(i) = FS(indx, i) - 1; % Calculate and store overshoot.
    % Go to where it crosses y = 1 the second time.
    indx = indx - 1 + find(FS(indx:end,i) <= 1, 1);
    width_end = indx; % Remember this location.
    % Go to the next local minimum (this is the undershoot).
    indx = indx - 1 + find(FS(indx+1:end,i) > FS(indx:end-1,i), 1);
    US(i) = 1 - FS(indx, i); % Calculate and store undershoot.
    % Calculate and store the width of the overshoot.
    W(i) = x(width_end) - x(width_start);
end

% Plot
close all;
figure(1);
hold on;
h = zeros(length(n)+1, 1); % Stores handles to curves
s = cell(length(n)+1, 1); % Stores legend strings
h(1) = plot(x, SW, 'k--');
s{1} = 'Square wave';
color = hsv(length(n)); % Generates different colors for each curve
for i=1:length(n)
    h(i+1) = plot(x, FS(:,i), 'Color', color(i,:));
    s{i+1} = sprintf('Partial sum with n = %d', n(i));
end
hold off;
xlim([-pi pi]);
xlabel('x');
title(['Square wave and partial sum of first n ', ...
       'non-zero terms of its Fourier series.']);
legend(h, s, 'Location', 'SouthEast');
```