18.085 Fall 2010 – HW 7 Solutions

3.5/1 Let f_{ijk} denote the values of a function $f(x,y,z)$ at the mesh points (hi, hj, hk), with $i,j,k = 1,\ldots,N$, $h = 1/(N+1)$. The discrete $3D$ Laplacian is given by

$$(-\Delta_{3\text{dscr}} u)_{ijk} := -h^{-2}(\Delta_i^2 u + \Delta_j^2 u + \Delta_k^2 u)_{ijk},$$

where $(\Delta_i^2 u)_{ijk} = u_{(i+1)jk} + u_{(i-1)jk} - 2u_{ijk}$ is the second difference in index i, and Δ_j^2, Δ_k^2 are the second differences in indices j,k respectively (we are assuming zero boundary conditions). Order all N^3 numbers u_{ijk} into one long $N^3 \times 1$ vector U so that

$$U_{N^2(i-1)+N(j-1)+k} = u_{ijk}.$$

Note that

$$-(\Delta_j^2 + \Delta_k^2)U =$$
$$\begin{pmatrix} (K2D)_N & & & \\ & (K2D)_N & & \\ & & \ddots & \\ & & & (K2D)_N \end{pmatrix} \begin{pmatrix} U(1:N^2) \\ U(N^2+1:2N^2) \\ \vdots \\ U(N^2(N-1)+1:N^3) \end{pmatrix}$$

where $(K2D)_N$ is the $N^2 \times N^2$ matrix that encodes the $2D$ discrete Laplacian. On the other hand,

$$-\Delta_i^2 U =$$
$$\begin{pmatrix} 2(I2D)_N & -(I2D)_N & & \cdot \\ -(I2D)_N & 2(I2D)_N & -(I2D)_N & \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & & -(I2D)_N & 2(I2D)_N \end{pmatrix} \begin{pmatrix} U(1:N^2) \\ U(N^2+1:2N^2) \\ \vdots \\ U(N^2(N-1)+1:N^3) \end{pmatrix}$$

where $(I2D)_N$ is the $N^2 \times N^2$ identity matrix. Thus,

$$(K3D)_N = \text{kron}(I_N, (K2D)_N) + \text{kron}(K_N, (I2D)_N).$$

3.6/2 The components of the load vector F

$$F_i = \int_0^1 \delta(x-a) V_i(x)\, dx = V_i(a).$$

If mh is the closest meshpoint to a which is less than or equal to a, then the only (possibly) non-zero entries of the load vector are

$$F_m = V_m(a) = 1 - \frac{a - mh}{h} \quad \text{and} \quad F_{m+1} = V_{m+1}(a) = \frac{a - mh}{h}.$$

3.6/3 Tetrahedrons are the obvious 3D analogues to triangles. Consider a tetrahedron whose vertices are

$$v_0 = (0,0,0) \quad v_1 = (1,0,0)h \quad v_2 = (0,1,0)h \quad v_3 = (0,0,1)h.$$

If $U_i = U(v_i)$ and U is a linear function of x, y, z:

$$U(x,y,z) = U_0 + \frac{x}{h}(U_1 - U_0) + \frac{y}{h}(U_2 - U_0) + \frac{z}{h}(U_3 - U_0).$$

3.6/7 Let us first find the function U in the lower triangle. $U(x,0) = a + bx + dx^2$ is a quadratic in x which is supposed to vanish for $x = 0, \frac{1}{2}, 1$. This happens only when $a = b = d = 0$. Similarly,

$$U(1,y) = a + b + d + (c+e)y + fy^2$$

is a quadratic that vanishes for $y = 0, \frac{1}{2}, 1$. So,

$$a + b + d = 0 \quad c + e = 0 \quad f = 0.$$

Thus, $U(x,y) = cy(1-x)$. From $U(1/2, 1/2) = 1$ we find $c = 4$:

$$U(x,y) = 4y(1-x) \qquad \text{in the lower triangle.}$$

Using the reflection symmetry about the diagonal $x = y$, one easily obtains

$$U(x,y) = 4x(1-y) \qquad \text{in the upper triangle.}$$

3.6/8 **(a)** A degree 3 polynomial in x, y has the form

$$p(x,y) = a_1 + (a_2 x + a_3 y) + (a_4 x^2 + a_5 xy + a_6 y^2) + (a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3).$$

The $p(x,y)$ that correspond to neighbouring triangles have to share the same values along the common side, because the test/trial functions have to be continuous.

(b) There are $4^2 = 16$ monomial terms $\{x^i y^j\}_{0 \le i,j \le 3}$ that span a bicubic polynomial. This number matches the 16 values of U, U_x, U_y, U_{xy} at the four vertices of a rectangle.

3.6/10 **(a)** A corner node R is shared by 4 square elements, so it interacts with all the nodes encompassed within those: there are 25 such nodes.

(b) A midpoint node M is shared by 2 square elements, so its neighbours are all the nodes within a 4×2 rectangle, centered at M: there are 15 of them.

(c) It's because the centre node interacts only with the nodes within its square element.

3.6/11 **(a)** If $v_1 = (1,1)$, $v_2 = (-1,1)$, $v_3 = (-1,-1)$ and $v_4 = (1,-1)$ and we want $U(v_i) = U_i$, then

$$\begin{aligned} U(x,y) &= U_1 \frac{(x+1)(y+1)}{4} + U_2 \frac{(x-1)(y+1)}{-4} \\ &+ U_3 \frac{(x-1)(y-1)}{4} + U_4 \frac{(x+1)(y-1)}{-4} \\ &= \frac{1}{4}((U_1+U_2+U_3+U_4) + (U_1-U_2-U_3+U_4)x \\ &+ (U_1+U_2-U_3-U_4)y + (U_1-U_2+U_3-U_4)xy) \end{aligned}$$

(b) From (a) we know

$$b = U_1 - U_2 - U_3 + U_4 \qquad c = U_1 + U_2 - U_3 - U_4.$$

Thus,

$$\begin{pmatrix} \partial U/\partial x \\ \partial U/\partial y \end{pmatrix}(0,0) = \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}.$$

(c) It's a straightforward check to see that $(1,1,1,1)^T$ and $(1,-1,1,-1)^T$ are in the nullspace of G.

3.6/18 The eigenvalue λ is the lowest eigenvalue of the matrix

$$M^{-1}K = \frac{18}{15}\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} = \frac{54}{5}\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} = \frac{54}{5}B$$

The characteristic polynomial of the matrix B is $p(t) = t^2 - 6t + 5$, whose roots are $t = 1, 5$. So, the lowest eigenvalue of $M^{-1}K$ is $\lambda = \frac{54}{5} = 10.8$ which is not too far from $\pi^2 \approx 9.87$.