# 18.085 Fall 2010 – HW 4 Solutions

**2.3/7** The coefficients $\hat{u} = (C, D)^T$ for the least squares fit satisfy the normal equation

$$A^T A\hat{u} = A^T b, \tag{1}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}, \qquad b = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

We compute both sides of (1) to get

$$\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \tag{2}$$

which we invert to obtain $(C, D) = (3, -1)$. Thus, the least squares fit to the given data is the line $y = 3 - x$.

**2.3/8** The projection of $b$ onto the linear subspace spanned by the columns of $A$ is given by $p = A\hat{u}$, so we compute $p = (3, 2, 1, 0)^T$ and obviously the points $(0, 3), (1, 2), (2, 1), (3, 0)$ lie on the line $y = 3 - x$. It's a straightforward check to verify that $e = b - p = (1, -1, -1, 1)^T$ satisfies $A^T e = 0$.

**2.3/9** We compute

$$E = \|b - Au\|^2 = (4-C)^2 + (1-(C+D))^2 + (0-(C+2D))^2 + (1-(C+3D))^2$$

so that

$$0 = -\frac{1}{2}\frac{\partial E}{\partial C} = (4 - C) + (1 - (C + D)) - (C + 2D) + (1 - (C + 3D)) = 6 - (4C + 6D)$$
$$0 = -\frac{1}{2}\frac{\partial E}{\partial D} = (1 - (C + D)) - 2(C + 2D) + 3(1 - (C + 3D)) = 4 - (6C + 14D)$$

and we recover the normal equation (2).

**2.3/18** The average $\hat{u}_{10}$ of the ten numbers $b_1, \dots, b_{10}$ satisfies

$$10\hat{u}_{10} = b_1 + \cdots + b_9 + b_{10} = 9\hat{u}_9 + b_{10},$$

so $\hat{u}_{10} = 9/10\hat{u}_9 + 1/10 b_{10}$.

**2.3/24** We would like to find the plane $b(x, y) = C + Dx + Ey$ that gives the best fit to the data $b = 0, 1, 3, 4$ and $(x, y) = (1, 0), (0, 1), (-1, 0), (0, -1)$ respectively. The normal equation $A^T A\hat{u} = A^T b$ for $\hat{u} = (C, D, E)^T$ takes the form

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ -3 \end{pmatrix}$$

so that $(C, D, E) = (2, -3/2, -3/2)$ and we check that

$$b(0, 0) = C = 2 = \mathrm{Av}(0, 1, 3, 4).$$

**2.4/3** The incidence matrix for the given network is the $5 \times 4$ matrix

$$A_{\mathrm{sq}} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Obviously $A_{\mathrm{sq}}(1, 1, 1, 1)^T = 0$. In the flow of currents picture a solution $w$ of $A^T w = 0$ is precisely a configuration of currents that obeys Kirchhoff's first law, i.e. the sum of the currents flowing through each node equals 0. Two linearly independent solutions that we can immediately read off from the network diagram are:

$$w_1 = (1, 0, -1, 1, 0)^T \qquad w_2 = (0, 1, -1, 0, 1)^T.$$

**2.4/7** The $3 \times 4$ incidence matrix that corresponds to the network in question is

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

and if $C = \mathrm{diag}(c_1, c_2, c_3)$ we compute

$$K = A^T C A = \begin{pmatrix} c_1 & 0 & 0 & -c_1 \\ 0 & c_2 & 0 & -c_2 \\ 0 & 0 & c_3 & -c_3 \\ -c_1 & -c_2 & -c_3 & c_1 + c_2 + c_3 \end{pmatrix}.$$

If we ground node 4 we form the reduced $K_{\mathrm{red}}$ by deleting the fourth row and the fourth column from $K$, so that $K_{\mathrm{red}} = \mathrm{diag}(c_1, c_2, c_3)$ and $\det K_{\mathrm{red}} = c_1 c_2 c_3$. Note that the unreduced $K$ is singular, so $\det K = 0$!

**2.4/8** Let $K_i$ be the $4 \times 4$ "element matrix" that corresponds to edge $i$ connecting node $j$ to node $k$, $j < k$. That is, $K_i$ is the matrix whose only non-zero entries are $(K_i)_{jj}, (K_i)_{jk}, (K_i)_{kj}, (K_i)_{kk}$ which equal $c_i, -c_i, -c_i, c_i$, respectively. Then

$$K = A^T C A = \sum_{\text{edges } i} K_i = \begin{pmatrix} c_1+c_2+c_4 & -c_1 & -c_2 & -c_4 \\ -c_1 & c_1+c_3+c_5 & -c_3 & -c_5 \\ -c_2 & -c_3 & c_2+c_3+c_6 & -c_6 \\ -c_4 & -c_5 & -c_6 & c_4+c_5+c_6 \end{pmatrix}.$$

When $C = I$, we get $(A^T A)_{jk} = K_{jk} = 3\delta_{jk} - 1(1-\delta_{jk})$.

**2.4/9** The matrix $A^T A = D - W$ where $D$ is a diagonal matrix whose $ii$-entry equals number of edges going through node $i$, and $W$ is the adjacency matrix associated with the network. So for the network which consists of 5 nodes connected in a line by 4 edges

$$A^T A = D - W = B_5.$$

Grounding node 5, we see that the reduced $K_{\text{red}} = T_4$, so that

$$(K_{\text{red}}^{-1})_{ij} = (T_4^{-1})_{ij} = 5 - \max(i,j).$$

Its determinant $\det K_{\text{red}} = \det T_4 = 1$ and its eigenvalues are

$$\lambda_k = 2 - 2\cos\left(\frac{k-\frac{1}{2}}{4+\frac{1}{2}}\pi\right) = \left\{2 - 2\cos\left(\frac{\pi}{9}\right), 1, 2-2\cos\left(\frac{5\pi}{9}\right), 2-2\cos\left(\frac{7\pi}{9}\right)\right\}.$$

**2.4/17** (a) The only non-zero entries of the matrix $A^T A = D - W$ are its diagonal entries and the off-diagonal ones that correspond to the presence of edges. Thus,

$$\#\text{ non-zero entries in } A^T A = 9 + 2 \times \#\text{ of edges} = 9 + 2 \times 12 = 33.$$

The number of the zero entries of $A^T A$ is then $9^2 - 33 = 48$.

(b) $D = \text{diag}(2,3,2,3,4,3,2,3,2)$.

(c) The four $-1$'s in the middle row of $-W$ correspond to the four other nodes that the center node connects to.

**2.4/18** We want to solve $Ku = f$ where $K$ is the $8 \times 8$ reduced matrix for the $3 \times 3$ grid from the previous problem (as we ground node $(3,3)$, we obtain $K$ by deleting the last row and column). The vector of current sources $f$ is the $8 \times 1$ vector whose first entry is 1 and all others are zero. Using MATLAB we derive $u(1,1) = u_1 = (K^{-1}f)_1 = 1.5$.