## 1.6: 16, 20, 24, 27, 28

16) If $A$ is positive definite, then $A^{-1}$ is positive definite.

The proof of the above statement can easily be shown for the following $2 \times 2$ matrix,

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

If that matrix is positive definite, then its upper left determinants must be positive. These conditions are stated by the inequalities $a > 0$ and $ac - b^2 > 0$. The inverse of this $2 \times 2$ matrix is easy to find; it is given by

$$A^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$$

For $A^{-1}$ to be positive definite, its upper left determinants must also be positive. The resulting inequalities can be simplified as shown by using the given inequality $ac - b^2 > 0$.

$$\frac{c}{ac - b^2} > 0 \qquad \frac{ca}{(ac-b^2)^2} - \frac{b^2}{(ac-b^2)^2} > 0$$

$$c > 0 \qquad \frac{(ac-b^2)}{(ac-b^2)^2} > 0$$

$$\frac{1}{ac - b^2} > 0$$

$$1 > 0$$

The inequality on the right is trivially satisfied. The inequality on the left, $c > 0$, is also automatically satisfied because if $c \leq 0$, it would imply that $b^2 < ac \leq 0$ (using the fact that $a > 0$), which is a contradiction because the the square of a real number cannot be strictly less than zero. Since both determinant tests for $A^{-1}$ are satisfied automatically, just by the fact that $A$ is positive definite, it shows that $A^{-1}$ is also positive definite given that $A$ is positive definite.

The case for a general matrix $A$ can also be proven, using the eigen decomposition $A = S\Lambda S^{-1}$. If $A$ is positive definite, then all of its eigenvalues must be positive, or the diagonal entries on the diagonal matrix $\Lambda$ are all positive. Since $\Lambda$ is a diagonal matrix, its inverse $\Lambda^{-1}$ must also be a diagonal matrix with its entries simply being the reciprocal of the corresponding entry in $\Lambda$; or mathematically, $[\Lambda^{-1}]_{ii} = ([\Lambda]_{ii})^{-1}$. This means that the diagonal entries of $\Lambda^{-1}$ are also positive. The inverse of $A$ is given by

$$A^{-1} = (S\Lambda S^{-1})^{-1} = (S^{-1})^{-1}\Lambda^{-1}S^{-1} = S\Lambda^{-1}S^{-1}$$

Since the diagonal entries of $\Lambda^{-1}$, which are known to be positive, are the eigenvalues of $A^{-1}$, it indicates that the eigenvalues of $A^{-1}$ are all positive, and thus $A^{-1}$ is positive definite.

20)

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Looking at the above equation, a few things are clear. First, the third matrix is the transpose of the first matrix. Second, the first matrix has orthonormal column vectors. This can easily be checked as follows: taking the inner product of the first column vector with itself gives $\cos^2\theta + \sin^2\theta = 1$; taking the inner product of the second column vector with itself gives $\sin^2\theta + \cos^2\theta = 1$; and taking the inner product of the first column vector with the second column vector gives $-\cos\theta\sin\theta + \sin\theta\cos\theta = 0$. Third, the second matrix is a diagonal matrix. Putting all these observations together it is clear that the right hand side is the eigen decomposition, $A = Q\Lambda Q^T$ where

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

With $A$ written in this form, the eigenvalues, eigenvectors, and determinant of $A$ are easy to calculate.

(a) The determinant of $A$ is the product of the eigenvalues, which is $2 \times 5 = 10$.

(b) The eigenvalues are on the diagonal of $\Lambda$ and they are $\lambda_1 = 2$ and $\lambda_2 = 5$.

(c) The eigenvectors are the column vectors of $Q$ and they are $v_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $v_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.

(d) Since the eigenvalues of $A$, given in part (b), are all positive, the matrix $A$ is positive definite.

24) We want to show that $P(u) = \frac{1}{2}u^TKu - u^Tf$ is equal to $\frac{1}{2}(u - K^{-1}f)^TK(u - K^{-1}f) - \frac{1}{2}f^TK^{-1}f$. We can start with the latter expression and simplify it to obtain the former expression. Note that in the simplification, the assumption that $K$ is a symmetric, positive definite matrix was used. Also, the fact that $u^Tf = f^Tu$ is used, which comes from the fact that the inner product of those two column vectors is a scalar and so the order of the inner product does not matter.

$$\begin{aligned}
\frac{1}{2}(u - K^{-1}f)^TK(u - K^{-1}f) - \frac{1}{2}f^TK^{-1}f &= \frac{1}{2}\left[u^TK(u - K^{-1}f) - f^T(K^T)^{-1}K(u - K^{-1}f)\right] - \frac{1}{2}f^TK^{-1}f \\
&= \frac{1}{2}\left[u^TKu - u^TKK^{-1}f - f^TK^{-1}Ku + f^TK^{-1}KK^{-1}f\right] - \frac{1}{2}f^TK^{-1}f \\
&= \frac{1}{2}[u^TKu] - \frac{1}{2}[u^Tf + f^Tu] + \frac{1}{2}f^TK^{-1}f - \frac{1}{2}f^TK^{-1}f \\
&= \frac{1}{2}u^TKu - u^Tf = P(u)
\end{aligned}$$

Other things to notice about P(u) can be seen by studying the longer expression representation. The term $\frac{1}{2}(u - K^{-1}f)^TK(u - K^{-1}f)$ can be simplified to $\frac{1}{2}v^TKv$ by letting $v = (u - K^{-1}f)$. Since K is positive definite, by the definition of positive definite, this term must always be positive for any $v \neq 0$. When $v$ does equal 0, it implies that $u = K^{-1}f$ or $Ku = f$, and under this case $P(u)$ simplifies to $-\frac{1}{2}f^TK^{-1}f$, which is the minimum energy $P_{\min}$.

27) We are given that the matrices $H$ (size $m \times m$) and $K$ (size $n \times n$) are positive definite and matrices $M$ and $N$ are defined in block notation by

$$M = \begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix} \qquad N = \begin{bmatrix} K & K \\ K & K \end{bmatrix}$$

If we denote the upper triangular Gaussian eliminated forms of $H$ and $K$ as $U_H$ and $U_K$ respectively, then we can perform Gaussian elimination on matrices $M$ and $N$ and get

$$M \xrightarrow{\text{elimination}} \begin{bmatrix} U_H & 0 \\ 0 & U_K \end{bmatrix} \qquad N \xrightarrow{\text{elimination}} \begin{bmatrix} U_K & U_K \\ 0 & 0 \end{bmatrix}$$

So the pivots of $M$ are composed of the pivots of $H$ and the pivots of $K$. Since the pivots of both $H$ and $K$ are positive, the pivots of $M$ are all positive and thus $M$ is positive definite. The pivots of $N$ are composed of the pivots of $K$ and $n$ zeros. Since $N$ has positive and zero pivots, it is not positive definite but rather positive semi-definite.

The eigenvalues of $M$ and $N$ can also be connected to the eigenvalues of $H$ and $K$. We define $v_i^H$ and $\lambda_i^H$ and to be the $m$ eigenvectors and corresponding eigenvalues of $H$ with $i = 1, 2, ..., m$. We also define $v_i^K$ and $\lambda_i^K$ and to be the $n$ eigenvectors and corresponding eigenvalues for $K$ with $i = 1, 2, ..., n$. Then the following observations can be made.

$$\begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} v_i^H \\ 0 \end{bmatrix} = \begin{bmatrix} Hv_i^H \\ 0 \end{bmatrix} = \lambda_i^H \begin{bmatrix} v_i^H \\ 0 \end{bmatrix} \qquad \begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} 0 \\ v_i^K \end{bmatrix} = \begin{bmatrix} 0 \\ Kv_i^K \end{bmatrix} = \lambda_i^K \begin{bmatrix} 0 \\ v_i^K \end{bmatrix}$$

So the eigenvalues of $M$ are composed of the eigenvalues of $H$ and the eigenvalues of $K$. Also if we define $e_i$ to be the column vector consisting of $(i - 1)$ zeros followed by a one and then followed by $(n - i)$ zeros, then we can use it to find the eigenvalues of $N$.

$$\begin{bmatrix} K & K \\ K & K \end{bmatrix}\begin{bmatrix} e_i \\ -e_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} e_i \\ -e_i \end{bmatrix} \qquad \begin{bmatrix} K & K \\ K & K \end{bmatrix}\begin{bmatrix} v_i^K \\ v_i^K \end{bmatrix} = \begin{bmatrix} 2Kv_i^K \\ 2Kv_i^K \end{bmatrix} = 2\lambda_i^K \begin{bmatrix} v_i^K \\ v_i^K \end{bmatrix}$$

Since $e_i$ is orthogonal to $e_j$ for $i \neq j$, it is clear that $\begin{bmatrix} e_i \\ -e_i \end{bmatrix}$ for $i = 1, 2, ..., n$ are $n$ linearly independent vectors. This means that zero is an eigenvalue for $N$ with a multiplicity of $n$. The remaining eigenvalues come from the eigenvalues of $K$, but as seen above they are doubled. So the eigenvalues of $N$ are 2 times the eigenvalues of $K$, as well as the eigenvalue zero with multiplicity $n$.

Finally, we want to construct the Cholesky of $M$, $\texttt{chol}(M)$, from $\texttt{chol}(H)$ and $\texttt{chol}(K)$. We let $A = \texttt{chol}(H)$, so that $H = A^T A$, and let $B = \texttt{chol}(K)$, so that $K = B^T B$. We then define a matrix $C$ that is given in block notation by

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} \texttt{chol}(H) & 0 \\ 0 & \texttt{chol}(K) \end{bmatrix}$$

If we multiply the transpose of C with C, we find that it equals

$$C^T C = \begin{bmatrix} A^T & 0 \\ 0 & B^T \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A^T A & 0 \\ 0 & B^T B \end{bmatrix} = \begin{bmatrix} H & 0 \\ 0 & K \end{bmatrix} = M$$

So, $M = C^T C$, which is the Cholesky factorization. Thus, $\texttt{chol}(M) = C$, where $C$ was defined above in terms of the $\texttt{chol}(H)$ and $\texttt{chol}(K)$.

28)

$$P = \begin{bmatrix} w_1 & w_2 & u \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix} = w_1^2 + w_2^2 - 2uw_1 + 2uw_2$$

The eigenvalues of the middle matrix, denoted $A$, are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = -1$, and their corresponding normalized eigenvectors are

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \qquad v_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$A$ can be written in eigen decomposed form, $A = Q\Lambda Q^T$, where

$$Q = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

So, $P = yAy^T$, where $y$ is defined to be $y = \begin{bmatrix} w_1 & w_2 & u \end{bmatrix}$. Then, $P = yAy^T = yQ\Lambda Q^T y^T$. Letting $x = Q^T y^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$, $P = x^T \Lambda x = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_2 x_3^2$. Then, $x$ is found as follows

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix} = \begin{bmatrix} \frac{w_1+w_2}{\sqrt{2}} \\ \frac{w_1-w_2-u}{\sqrt{3}} \\ \frac{w_1-w_2+2u}{\sqrt{6}} \end{bmatrix}$$

So $P = x^T \Lambda x = 1 \left(\frac{w_1+w_2}{\sqrt{2}}\right)^2 + 2\left(\frac{w_1-w_2-u}{\sqrt{3}}\right)^2 - 1\left(\frac{w_1-w_2+2u}{\sqrt{6}}\right)^2$. The expression on the right can be simplified considerably.

$$\begin{aligned}
P &= 1\left(\frac{w_1+w_2}{\sqrt{2}}\right)^2 + 2\left(\frac{w_1-w_2-u}{\sqrt{3}}\right)^2 - 1\left(\frac{w_1-w_2+2u}{\sqrt{6}}\right)^2 \\
&= \frac{1}{2}(w_1+w_2)^2 + \frac{2}{3}\left[\underline{\underline{(w_1-w_2)^2}} - \underline{2(w_1-w_2)u} + \cancel{u^2}\right] - \frac{1}{6}\left[\underline{\underline{(w_1-w_2)^2}} + \underline{4(w_1-w_2)u} + \cancel{4u^2}\right] \\
&= \frac{1}{2}(w_1+w_2)^2 + \frac{1}{2}(w_1-w_2)^2 - 2(w_1-w_2) \\
&= \frac{1}{2}(w_1^2+w_2^2+\cancel{2w_1w_2}) + \frac{1}{2}(w_1^2+w_2^2-\cancel{2w_1w_2}) - 2w_1u + 2w_2u \\
&= w_1^2 + w_2^2 - 2uw_1 + 2uw_2
\end{aligned}$$

This agrees with the original given equation.

## 2.1: 5, 6, 7, 8

5) In the case of four identical springs connecting three identical masses together and to the fixed top and bottom, the matrix $K$ relating the displacements of the masses, $u$, to the forces on the masses, $f$, with the equation $Ku = f$, is actually the $3 \times 3$ special $K$ matrix times the spring constant, $c$. The force on each mass is identical since it is simply equal to $mg$. Note that the convention here is to take both displacements and forces in the downward direction to be positive. From the derivation of $K$ in the fixed-fixed case it was found that $K = ACA^T$, where in this case $C = cI$, and $A$ along with the other matrices are listed below.

$$K = c\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{bmatrix} \qquad f = mg\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The tension in the springs is given by $w = CAu = CA(K^{-1}f)$.

$$CA = c\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad K^{-1} = \frac{1}{4c}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad CAK^{-1}f = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ -1 & 2 & 1 \\ -1 & -2 & 1 \\ -1 & -2 & -3 \end{bmatrix} mg\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{mg}{4}\begin{bmatrix} 6 \\ 2 \\ -2 \\ -6 \end{bmatrix}$$

So, the reaction force at the top (where pointing downward is positive) is $R_t = -w_1 = -\frac{3}{2}mg$. The reaction force at the bottom (where again pointing downward is positive) is $R_b = w_4 = -\frac{3}{2}mg$. Both reaction forces are negative meaning they are both pointing upward, to counteract the force of gravity pointing downward. Also, notice that the sum of the reaction forces $R_t + R_b = 3mg$, which is to be expected because the reaction forces must exactly balance out the force of gravity on the three masses.

6) Now, in the fixed-free case with three equal masses and three springs, the first and third springs have spring constant $c_1 = c_3 = 1$, but the second spring constant, $c_2$, differs. The matrix $K$ in $Ku = f = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ is given by the following equation

$$K = A^TCA = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} = \begin{bmatrix} 1 + c_2 & -c_2 & 0 \\ -c_2 & 1 + c_2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

For $c_2 = 10$:

$$K = \begin{bmatrix} 11 & -10 & 0 \\ -10 & 11 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad u = K^{-1}f = \begin{bmatrix} 3.0 \\ 3.2 \\ 4.2 \end{bmatrix}$$

For $c_2 = 100$:

$$K = \begin{bmatrix} 101 & -100 & 0 \\ -100 & 101 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad u = K^{-1}f = \begin{bmatrix} 3.00 \\ 3.02 \\ 4.02 \end{bmatrix}$$

7) In the fixed-fixed case with three equal masses and originally four equal springs with spring constant equal to 1, we now weaken spring 2 so that $c_2 \to 0$. Now, the $K$ matrix becomes

$$K = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

This matrix is still invertible because its determinant is 1. Solving $Ku = f = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, we get $u = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}^T$. To explain this answer physically, it is important to realize that by weaken spring 2, it splits the problem into two decoupled problem. One problem is mass 1 hanging freely off spring 1. In this problem, we expect for the displacement to be $u_1 = 1$ because there is a force of 1 on the mass connected to a spring with spring constant equal to 1. The second problem is a free-fixed problem with two identical masses and two identical spring, but the problem is upside compared to the typical fixed-free spring-mass problem. So, when $u_2 = 3$ and $u_3 = 2$, it means that spring 4 is compressed by 2 by the two masses above it and spring 3 is compressed by 1 by the one mass above it, which makes sense physically.

8) With one free-free spring, the extension in the spring is $e = u_2 - u_1$. The tension is proportional to the extension with the constant of proportionality, $c$, which is the spring constant. So, the tension $w = c \begin{bmatrix} -1 & 1 \end{bmatrix} u$. Then the tension is related to two forces at the ends of the spring, $f_i^t$ for the force at the top and $f_i^b$ for the force on bottom, by $f_i^t = -w$ and $f_i^b = w$. So,

$$f = \begin{bmatrix} f_i^t \\ f_i^b \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} w = c \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} u = c \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u$$

$$K_{\text{elem},i} u = f \qquad \text{where} \qquad K_{\text{elem},i} = c_i \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The notation used on the bottom is to distinguish between the various forces when assembling multiple element matrices, $K_{\text{elem},i}$, into a bigger matrix $K$ for the whole system. The superscript denotes whether the force is on top of the spring, $t$, or on the bottom of the spring, $b$. The subscript is supposed to be a number to distinguish which element matrix the forces are meant for. Also, as an example, if two springs, spring 1 and spring 2, are attached to the same mass with spring 1 above spring 2, then from a free-body diagram on the mass it can be seen that $f_1^b + f_2^t = f_m$, where $f_m$ is the total force on the mass.

(a) The element matrices can be assembled to find $K_{\text{free-free}}$ for the free-free three-mass, two-spring (spring 2 and spring 3) problem.

$$\begin{bmatrix} f_2^t \\ f_2^b \end{bmatrix} = c_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \xrightarrow{\text{expanded}} \begin{bmatrix} f_2^t \\ f_2^b \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} f_3^t \\ f_3^b \end{bmatrix} = c_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \xrightarrow{\text{expanded}} \begin{bmatrix} 0 \\ f_3^t \\ f_3^b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\text{Summing the two equations:} \quad \begin{bmatrix} f_2^t \\ f_2^b + f_3^t \\ f_3^b \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \underbrace{\begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & (c_2 + c_3) & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}}_{K_{\text{free-free}}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{1}$$

(b) Now, an element matrix for spring 1 can be adjusted and added to $K_{\text{free-free}}$ to get $K_{\text{fixed-free}}$, which is the matrix for the fixed-free three-mass, three-spring problem.

$$\begin{bmatrix} f_1^t \\ f_1^b \end{bmatrix} = c_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ u_1 \end{bmatrix} \xrightarrow{\text{expanded}} \begin{bmatrix} f_1^t \\ f_1^b \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 & -c_1 & 0 & 0 \\ -c_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} \xrightarrow{\text{simplified}} \begin{bmatrix} f_1^b \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Adding this equation to equation 1 gives

$$\begin{bmatrix} f_1^b + f_2^t \\ f_2^b + f_3^t \\ f_3^b \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \underbrace{\begin{bmatrix} (c_1 + c_2) & -c_2 & 0 \\ -c_2 & (c_2 + c_3) & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}}_{K_{\text{fixed-free}}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{2}$$

where this time $f_1$ has a different equation because adding spring 1 changes the free-body diagram of mass 1. Looking at the free-body diagram it can be seen that $f_1 = f_1^b + f_2^t$ is consistent with balancing the forces on mass 1.

(c) Finally, an element matrix for spring 4 can be adjusted and added to $K_{\text{fixed-free}}$ to get $K_{\text{fixed-fixed}}$, which is the matrix for the fixed-fixed three-mass, four-spring problem.

$$\begin{bmatrix} f_4^t \\ f_4^b \end{bmatrix} = c_4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_3 \\ 0 \end{bmatrix} \xrightarrow{\text{expanded}} \begin{bmatrix} 0 \\ 0 \\ f_4^t \\ f_4^b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_4 & -c_4 \\ 0 & 0 & -c_4 & c_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \end{bmatrix} \xrightarrow{\text{simplified}} \begin{bmatrix} 0 \\ 0 \\ f_4^t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Adding this equation to equation 2 gives

$$\begin{bmatrix} f_1^b + f_2^t \\ f_2^b + f_3^t \\ f_3^b + f_4^t \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \underbrace{\begin{bmatrix} (c_1 + c_2) & -c_2 & 0 \\ -c_2 & (c_2 + c_3) & -c_3 \\ 0 & -c_3 & (c_3 + c_4) \end{bmatrix}}_{K_{\text{fixed-fixed}}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{3}$$

and again $f_3$ has a different equation because adding spring 3 changes the free-body diagram of mass 3.

## 2.2: 5, 6, 8

5)

$$\frac{du}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} u$$

(a) $||u(1)||^2 = u_1^2 + u_2^2 + u_3^2$, and the derivative of $||u(t)||^2$ with respect to time is $2u_1u_1' + 2u_2u_2' + 2u_3u_3'$. This can be simplified as follows.

$$\begin{aligned} \frac{d}{dt}||u(t)||^2 &= 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2) \\ &= 2cu_1u_2 - 2bu_1u_3 + 2au_2u_3 - 2cu_1u_2 + 2bu_1u_3 - 2au_2u_3 = 0 \end{aligned}$$

Thus $||u(t)||^2 = ||u(0)||^2$.

(b) $Q = e^{At}$ is an orthogonal matrix, where $A$ is the matrix above. It can also be seen by studying the matrix that $A$ is a skew-symmetric matrix, or $A^T = -A$. It can be shown that $Q^T = e^{-At}$, and thus $Q^TQ = e^{-At}e^{At} = I$, satisfying the property for an orthogonal (actually orthonormal) matrix. The proof below makes use of the Taylor series expansion of $e^{At}$ about $t = 0$.

$$\begin{aligned} Q &= e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + ... \\ Q^T &= \left[I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + ...\right]^T = I + A^Tt + \frac{1}{2!}(A^Tt)^2 + \frac{1}{3!}(A^Tt)^3 + ... \\ &= I + (-At) + \frac{1}{2!}(-At)^2 + \frac{1}{3!}(-At)^3 + ... = e^{-At} \end{aligned}$$

6) The trapezoidal rule for $u' = Au$ is given by

$$(I - \frac{\Delta t}{2}A)U_{n+1} = (I + \frac{\Delta t}{2}A)U_n$$

If $A^T = -A$, then the trapezoidal rule will conserve the energy $||u||^2$. This can be proven by showing that $||U_{n+1}||^2 = ||U_n||^2$.

$$\begin{aligned} (I - \frac{\Delta t}{2}A)U_{n+1} &= (I + \frac{\Delta t}{2}A)U_n \\ U_{n+1} - U_n &= \frac{\Delta t}{2}A(U_{n+1} + U_n) \\ (U_{n+1}^T + U_n^T)(U_{n+1} - U_n) &= (U_{n+1}^T + U_n^T)\frac{\Delta t}{2}A(U_{n+1} + U_n) \\ U_{n+1}^TU_{n+1} + \cancel{U_{n+1}^TU_n} - \cancel{U_n^TU_{n+1}} - U_n^TU_n &= \frac{\Delta t}{2}\left[(U_{n+1} + U_n)^TA(U_{n+1} + U_n)\right] \\ ||U_{n+1}||^2 - ||U_n||^2 &= \frac{\Delta t}{2}v^TAv \end{aligned}$$

where we let $v = (U_{n+1} + U_n)$ and $U_{n+1}^TU_n$ was canceled by $U_n^TU_{n+1}$ since they are both equal scalars and the order of the inner product does not matter. Likewise, $v^TAv$ is a scalar and thus it is equal to its transpose $(v^TAv)^T = v^TA^Tv$. However, since $A^T = -A$, this means that $v^TAv = -v^TAv$ and the only way that equality can be satisfied is if $v^TAv = 0$. Thus,

$$\begin{aligned} ||U_{n+1}||^2 - ||U_n||^2 &= \frac{\Delta t}{2}v^TAv = 0 \\ ||U_{n+1}||^2 &= ||U_n||^2 \end{aligned}$$

8) The Forward and Backward Euler are given by

| Forward Euler | Backward Euler |
|---|---|
| $U_{n+1} = U_n + hV_n$ | $U_{n+1} = U_n + hV_{n+1}$ |
| $V_{n+1} = V_n - hU_n$ | $V_{n+1} = V_n - hU_{n+1}$ |

Forward Euler multiplies the energy by $(1 + h^2)$ at every step.

$$\begin{aligned} U_{n+1}^2 + V_{n+1}^2 &= (U_n + hV_n)^2 + (V_n - hU_n)^2 = U_n^2 + \cancel{2hU_nV_n} + h^2V_n^2 + V_n^2 - \cancel{2hU_nV_n} + h^2U_n^2 \\ &= (1 + h^2)U_n^2 + (1 + h^2)V_n^2 = (1 + h^2)(U_n^2 + V_n^2) \end{aligned}$$

Backward Euler divides the energy by $(1 + h^2)$ at every step. First, we rewrite the Backward Euler as $U_{n+1} - hV_{n+1} = U_n$ and $V_{n+1} + hU_{n+1} = V_n$. Then, we sum $U_n^2$ and $V_n^2$ to find

$$\begin{aligned} U_n^2 + V_n^2 &= (U_{n+1} - hV_{n+1})^2 + (V_{n+1} + hU_{n+1})^2 \\ U_n^2 + V_n^2 &= U_{n+1}^2 - \cancel{2hU_{n+1}V_{n+1}} + h^2V_{n+1}^2 + V_{n+1}^2 + \cancel{2hU_{n+1}V_{n+1}} + h^2U_{n+1}^2 \\ U_n^2 + V_n^2 &= (1 + h^2)U_{n+1}^2 + (1 + h^2)V_{n+1}^2 \\ U_n^2 + V_n^2 &= (1 + h^2)(U_{n+1}^2 + V_{n+1}^2) \\ U_{n+1}^2 + V_{n+1}^2 &= \frac{1}{1 + h^2}(U_n^2 + V_n^2) \end{aligned}$$

We can see what the gain in energy will be after 32 steps, for example, with $h = \frac{2\pi}{32}$: $(1 + h^2)^{32} \approx 3.355$. It is interesting to see whether Euler converges, so we take the limit of $y = (1 + h^2)^{\frac{2\pi}{h}}$ as $h \to 0$.

$$\begin{aligned} \lim_{h\to 0}(1 + h^2)^{\frac{2\pi}{h}} &= \lim_{h\to 0} y = \lim_{h\to 0} e^{\ln y} = \exp\left[\lim_{h\to 0} \ln y\right] = \exp\left[\lim_{h\to 0} \frac{2\pi}{h} \ln(1 + h^2)\right] \\ &= \exp\left[2\pi \lim_{h\to 0} \frac{\frac{d}{dh}\ln(1 + h^2)}{\frac{d}{dh}h}\right] = \exp\left[2\pi \lim_{h\to 0} \frac{2h}{1 + h^2}\right] = e^0 = 1 \end{aligned}$$

Since the limit equals 1, it indicates that Euler does converge, albeit slowly.

## MATLAB Assignment

In this problem, 100 identical masses are connected by identical springs with spring constant $c = 1$. Two boundary conditions are considered: one is the fixed-fixed boundary condition, in which the top and bottom masses are attached with springs to fixed supports; the other is the fixed-free boundary condition, in which only the top mass is attached with a spring to a fixed support and the bottom mass is hanging freely. The force on each mass is taken to have a magnitude of 0.01. MATLAB was used to generate the solutions to these problems and graph the displacements. The code used is displayed below in Listing 1. The plot of the displacement for the fixed-fixed case can be seen in Figure 1. The plot of the displacement for the fixed-free case can be seen in Figure 2.

Listing 1: MATLAB Code to solve spring-mass system for both boundary conditions and plot displacements.

```
% Prepare necessary matrices for system under both boundary conditions
K = KTBC('K', 100, 1); % Create sparse K matrix for fixed-fixed case
H = K; H(100, 100) = 1; % Create sparse H matrix for fixed-free case
f = 0.01 * ones(100, 1); % Create force column vector

% Solve system Ku = f for both boundary conditions
u1 = K\f; % Solve displacements, u1, for fixed-fixed case
u2 = H\f; % Solve displacements, u2, for fixed-free case

% Plot results
figure(1);
plot(u1, '+'); % Plot results for fixed-fixed case
xlabel('Mass Number');
ylabel('Displacement');
title('Fixed-Fixed Case');

figure(2);
plot(u2, '+'); % Plot results for fixed-free case
xlabel('Mass Number');
ylabel('Displacement');
title('Fixed-Free Case');
```

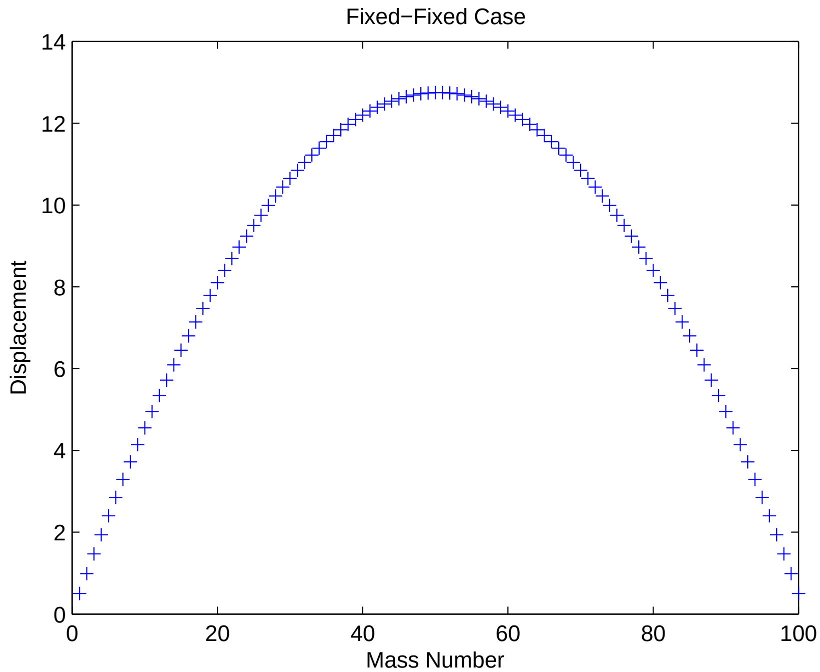


Figure 1: Plot of mass displacement for spring-mass system with a fixed-fixed boundary condition.

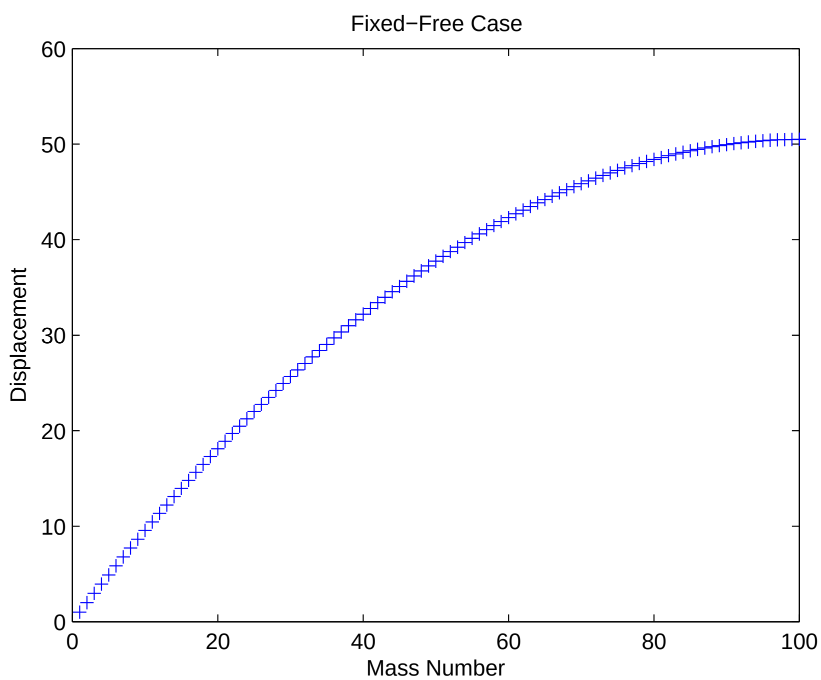


Figure 2: Plot of mass displacement for spring-mass system with a fixed-free boundary condition.