1.2: 1

1) The second derivate, $u''(x)$, of

$$u(x) = \begin{cases} Ax & \text{if } x \le 0 \\ Bx & \text{if } x \ge 0 \end{cases}$$

can be found by taking the derivative twice, and it is given by

$$u'(x) = \begin{cases} A & \text{if } x \le 0 \\ B & \text{if } x \ge 0 \end{cases}$$

$$\boxed{u''(x) = (B - A)\delta(x)}$$

The second difference, $\Delta^2 U_n$, of

$$U_n = \begin{cases} An & \text{if } n \le 0 \\ Bn & \text{if } n \ge 0 \end{cases} = \begin{bmatrix} \vdots \\ -2A \\ -A \\ 0 \\ B \\ 2B \\ \vdots \end{bmatrix}$$

can be found using the definition of the second difference

$$\Delta^2 U_n = U_{n+1} - 2U_n + U_{n-1}$$

The second difference is calculated for the following three difference cases.

if $n < 0$: $\quad \Delta^2 U_n = A(n+1) - 2A(n) + A(n-1) = 0$

if $n > 0$: $\quad \Delta^2 U_n = B(n+1) - 2B(n) + B(n-1) = 0$

if $n = 0$: $\quad \Delta^2 U_n = B(1) - 2(0) + A(-1) = B - A$

So, the second difference is written as

$$\boxed{\Delta^2 U_n = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ (B-A) \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{cases} 0 & \text{if } n < 0 \\ (B-A) & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}}$$

1.3: 1, 9, 18

1) We are interested in factorizing the K_4 matrix into $K_4 = LDL^T$. First, we must find the LU factorization of K_4 through Gaussian elimination.

$$K_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{l_{21}=-\frac{1}{2}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{l_{32}=-\frac{2}{3}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \longrightarrow$$

$$\xrightarrow{l_{43}=-\frac{3}{4}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} = U = DL^T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} L^T$$

L is found through the multiplying factors, l_{ij}, used in the Gaussian elimination. $[L]_{ij} = l_{ij}$, where l_{ij} is the factor that multiplies row j to get the new row that once subtracted from row i does the proper elimination of the leading non-zero element of that row. The non-zero l_{ij} were listed above when carrying out the elimination.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix}$$

And finally, K_4 is written in the LDL^T factorization,

$$\boxed{K_4 = LDL^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & 0 & -3/4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & -3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

The determinant of K_4 is now easy to find because the determinant of the product of matrices is the product of the determinant of each matrix, also the determinant of L is 1 and the determinant of a diagonal matrix is simply the product of the elements on the diagonal. So, $\det(K_4) = \det(L)\det(D)\det(L^T) = (1)[(2)(3/2)(4/3)(5/4)](1) = \boxed{\det(K_4) = 5}$.

9) We are interested in Cholesky factorizing the matrices K_3, T_3, and B_3 using the MATLAB command `chol`. $A = \texttt{chol}(K)$ produces an upper triangular matrix A such that $K = A^T A$.

$$\texttt{chol}(K_3) = \texttt{chol}\left(\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \right) = \begin{bmatrix} \sqrt{2} & \sqrt{1/2} & 0 \\ 0 & \sqrt{3/2} & \sqrt{2/3} \\ 0 & 0 & \sqrt{4/3} \end{bmatrix}$$

$$\texttt{chol}(T_3) = \texttt{chol}\left(\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\texttt{chol}(B_3)$ fails because B_3 is not a positive definite matrix, and the Cholesky factorization only works on positive definite matrices. To get around this problem we add the identity matrix multiplied by a small factor ϵ ($0 < \epsilon \ll 1$), using the MATLAB command `eps*eye(3)`, to the matrix B_3 and try again.

$$\texttt{chol}(B_3 + \epsilon I_3) = \texttt{chol}\left(\begin{bmatrix} 1+\epsilon & -1 & 0 \\ -1 & 2+\epsilon & -1 \\ 0 & -1 & 1+\epsilon \end{bmatrix} \right) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

18) A 3×3 matrix, A, is desired which has rows r_1, r_2, and r_3 such that $r_1 - 2r_2 + r_3 = 0$. An example of such a matrix is given below.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Now, if we designate the columns of A as c_1, c_2, and c_3, we wish to find a linear combination of these column vectors that sums to zero, similar to the condition on the rows above. Specifically, we want to find k_1, k_2, and k_3 that satisfies the equation $k_1c_1 + k_2c_2 + k_3c_3 = 0$. In turns out that for the example matrix A given above, one possible set of solutions is $k_1 = -1$, $k_2 = -2$, and $k_3 = 1$, or

$$-c_1 - 2c_2 + c_3 = 0$$

1.4: 1, 4, 7, 11

1) For $-u'' = \delta(x - a)$ with boundary conditions $u(0) = 2$ and $u(1) = 0$, the solution is given as

$$u(x) = \begin{cases} Ax + B & \text{for } 0 \le x \le a \\ Cx + B & \text{for } a \le x \le 1 \end{cases}$$

where A, B, C, and D are constants which we can determine by finding four independent conditions involving these constants.

The first two conditions are found from the boundary conditions.

$$u(0) = \boxed{B = 2}$$

$$u(1) = \boxed{C + D = 0}$$

The third condition is found by realizing that since $u(x)$ is continuous everywhere in the interval including at $x = a$, then $u(a)$ approached from the left side, $u(a^-) = Aa + B$, must be equal to $u(a)$ approached from the right side, $u(a^+) = Ca + D$.

$$\boxed{Aa + B = Ca + D}$$

The fourth, and final, condition is found by realizing that there is a drop of magnitude 1 in $u'(x)$ at $x = a$; this can be seen by integrating the differential equation and getting $u'(x) = -S(x - a) + \text{constant}$, where $S(x)$ is the step function. So, $u'(a^-) - u'(a^+) = 1$. For $0 \le x \le a$, $u'(x) = A$ and so $u'(a^-) = A$, and for $a \le x \le 1$, $u'(x) = C$ and so $u'(a^+) = C$. This leads to the fourth condition,

$$\boxed{A - C = 1}$$

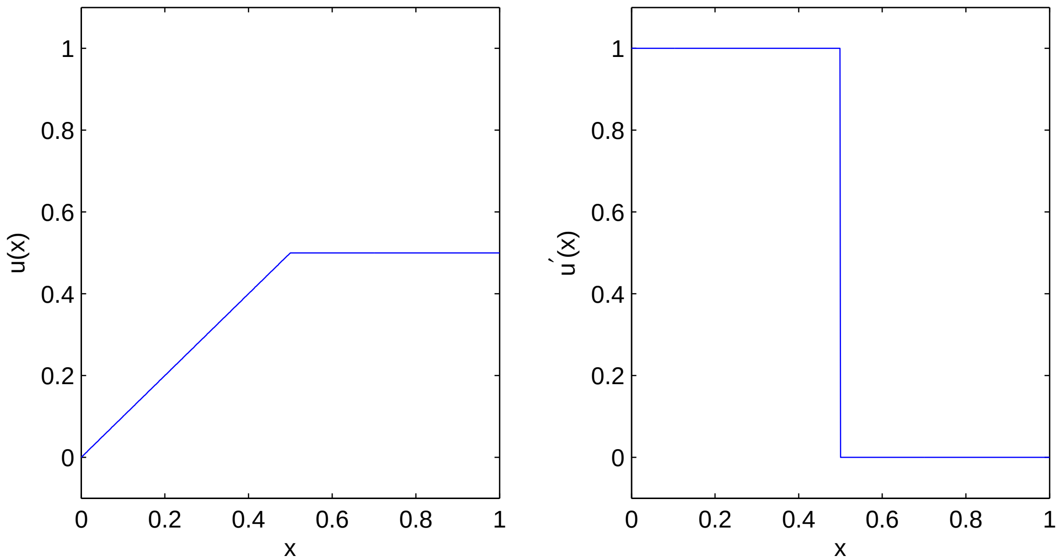
4) We solve $-d^2u/dx^2 = \delta(x - a)$ with the fixed-free boundary conditions $u(0) = 0$ and $u'(1) = 0$, by integrating both sides twice and simplifying using the boundary conditions when possible. It is assumed that $0 \le a \le 1$.

$$\begin{aligned} -\int_x^1 u''(\tilde{x})d\tilde{x} &= \int_x^1 \delta(\tilde{x} - a)d\tilde{x} \\ -u'(1) + u'(x) &= S(1 - a) - S(x - a) \\ 0 + \int_0^x u'(\tilde{x})d\tilde{x} &= \int_0^x (1 - S(\tilde{x} - a))\, d\tilde{x} \\ u(x) - u(0) &= x - R(x - a) + R(0 - a) \\ u(x) - 0 &= x - R(x - a) + 0 \end{aligned}$$

So, the solution $u(x)$ is

$$\boxed{u(x) = x - R(x - a) = \begin{cases} x & \text{for } 0 \le x \le a \\ a & \text{for } a \le x \le 1 \end{cases}}$$

The graphs of $u(x) = x - R(x-a)$ and $u'(x) = 1 - S(x-a)$ are shown below for the case where $a = 0.5$.



7) We solve $-u''(x) = f(x) = \delta(x - 1/3) - \delta(x - 2/3)$ with the free-free boundary conditions $u'(0) = 0$ and $u'(1) = 0$. We integrate both sides and simplify with the boundary conditions.

$$\begin{aligned} -\int_0^x u''(\tilde{x})d\tilde{x} &= \int_0^x \left[\delta(\tilde{x} - 2/3) - \delta(\tilde{x} - 1/3)\right] d\tilde{x} \\ u'(x) - u'(0) &= S(x - 2/3) - S(-2/3) - S(x - 1/3) + S(-1/3) \\ u'(x) - 0 &= S(x - 2/3) - 0 - S(x - 1/3) + 0 \end{aligned}$$

We can check to make sure $u'(1) = 0$.

$$u'(1) = S(1 - 2/3) - S(1 - 1/3) = S(1/3) - S(2/3) = 1 - 1 = 0$$

We integrate both sides again to get $u(x)$.

$$\begin{aligned} \int_0^x u'(\tilde{x})d\tilde{x} &= \int_0^x \left[S(\tilde{x} - 2/3) - S(\tilde{x} - 1/3)\right] d\tilde{x} \\ u(x) - u(0) &= R(x - 2/3) - R(-2/3) - R(x - 1/3) + R(-1/3) \\ u(x) &= R(x - 2/3) - 0 - R(x - 1/3) + 0 + u(0) \end{aligned}$$

And so, we finally find an equation for $u(x)$,

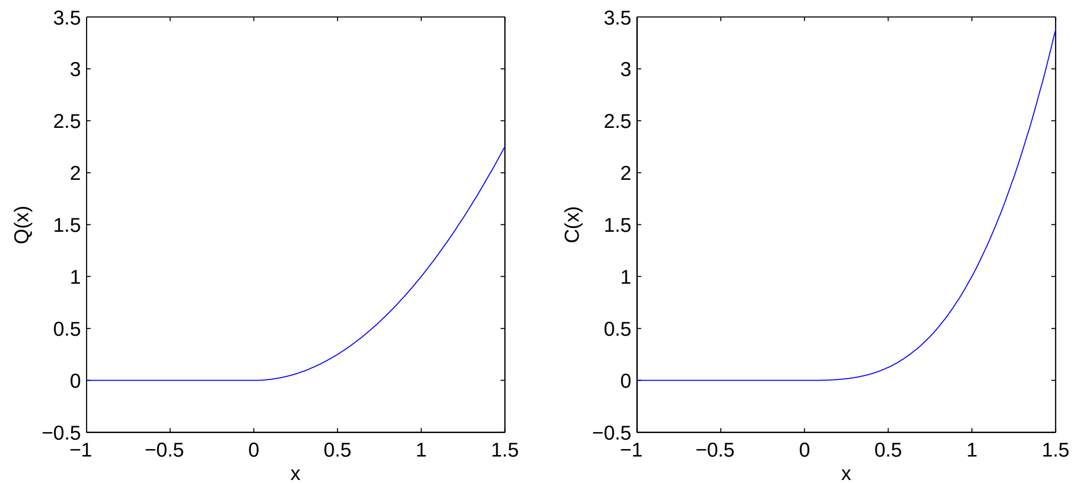
$$\boxed{u(x) = R(x - 2/3) - R(x - 1/3) + C}$$

where $C = u(0)$ is any arbitrary constant. Notice that since C can be any constant, there are infinitely many solutions $u(x)$ to the differential equation.

11) We already know that the integral of $\delta(x)$ is the step function $S(x)$, and the integral of $S(x)$ is the ramp function $R(x)$. Going further, the integral of $R(x)$ is the quadratic spline $Q(x)$, and the integral of $Q(x)$ is the cubic spline $C(x)$. The formulas for $S(x)$, $R(x)$, $Q(x)$, and $C(x)$ are written below.

$$S(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases} \quad R(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases} \quad Q(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2}x^2 & \text{for } x \geq 0 \end{cases} \quad C(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{6}x^3 & \text{for } x \geq 0 \end{cases}$$

$Q(x)$ and $C(x)$ are plotted below for $-1 \le x \le 1.5$.



The first and second derivatives of $C(x)$ are continuous at $x = 0$, while only the first derivative of $Q(x)$ is continuous at $x = 0$.

1.5: 3, 4, 8, 9, 13

3) We want to find the eigenvalues of K_5, and verify that they equal $(2-\sqrt{3}, 2-1, 2-0, 2+1, 2+\sqrt{3})$. This is done using MATLAB. First the decimal values for the eigenvalues are found using the MATLAB command `e = eig(K)`. We can compare these numbers with the numbers for the eigenvalues generated with the formula $\lambda_k = 2 - \cos(\frac{k\pi}{n+1})$, where $k = 1, 2, ..., n$ and in this case $n = 5$. These values are generated with the MATLAB command `e_expected = 2*ones(5,1) - 2*cos([1:5]*pi/6)'`. Taking the difference between `e` and `e_expected` in MATLAB, we get a column of zeros (within a tolerance of 1.0×10^{-15}), indicating that the two are equal.

4) We want to generate the Discrete Sine Transform (DST) in two ways. First, we want to construct DST by using the eigenvector matrix, Q, found using the MATLAB command `[Q,E] = eig(K)` where `K` refers to the 5×5 special matrix K. We also change the sign of a few of the columns so that the first element on each column is positive. The second way to generate DST is by directly using the formula $[\text{DST}]_{jk} = \sin(\frac{jk\pi}{n+1})$. The MATLAB code below generates DST using both methods and compares them. Either method generated the same DST within tolerance. Also, the MATLAB code below demonstrated that $[\text{DST}]^T = [\text{DST}]^{-1}$ within tolerance.

```
% Generate DST using eigenvectors of K for the n = 5 case.
K = KTBC('K', 5); [Q, E] = eig(K); % KTBC function is available on the class website.
DST1 = Q * diag([-1 -1  1 -1  1]); % Adjust sign of columns

% Generate DST using the given formula for the n = 5 case.
JK =  [1:5]' * [1:5]; % 5x5 matrix where the entry in row j, column k is j*k
DST2 = sin(JK * pi/6) / sqrt(3); % Follow DST formula but then also normalize each column
% Why sqrt(3)? If you take the norm of each column vector you will get sqrt(3) for each
% one. This can easily be seen by doing (sin(JK*pi/6))' * (sin(JK*pi/6)) and getting 3*I.

% Compare and check
% Note: max(max(abs(A-B))) finds the largest magnitude difference between the
%       corresponding elements  of the matrices A and B.
largest_difference1 = max(max(abs(DST2 - DST1)))  % Less than 1e-15 (within tolerance)
largest_difference2 = max(max(abs(DST1' - inv(DST1)))) % Less than 1e-15 (within tolerance)
```

8) The n eigenvalues of K_n are given by $\lambda_k = 2 - 2\cos\frac{k\pi}{n+1}$. The sum of these n eigenvalues must equal the trace of K_n, which is simply $2n$. The following shows this equality by proving that $\sum_{k=1}^{n}\lambda_k = 2n$.

$$\begin{aligned}\sum_{k=1}^{n}\left(2 - 2\cos\left(\frac{k\pi}{n+1}\right)\right) &= \sum_{k=1}^{n}(2) - \sum_{k=1}^{n}\left(\exp\left(\frac{ik\pi}{n+1}\right) + \exp\left(\frac{-ik\pi}{n+1}\right)\right)\\ &= 2n + \sum_{k=0}^{n}\left(\exp\left(\frac{ik\pi}{n+1}\right) + \exp\left(\frac{-ik\pi}{n+1}\right)\right) - 2\end{aligned}$$

In order, to show the equality holds, we now need to show that the summation in the righthand side of the above equation sums to 2. To show that, we first define $r = \exp\left(\frac{i\pi}{n+1}\right)$. Also note that, $r^{(n+1)} = e^{i\pi} = -1$. Now the proof continues by evaluating the summation and demonstrating that it equals 2.

$$\begin{aligned}\sum_{k=0}^{n}\left(\exp\left(\frac{ik\pi}{n+1}\right) + \exp\left(\frac{-ik\pi}{n+1}\right)\right) &= \sum_{k=0}^{n}\left(r^k + \left(\frac{1}{r}\right)^k\right) = \frac{1 - r^{(n+1)}}{1-r} + \frac{1 - \left(\frac{1}{r}\right)^{(n+1)}}{1 - \frac{1}{r}} \quad \text{(using geometric series)}\\ &= \frac{r^n}{r^n}\left(\frac{1 - r^{(n+1)}}{1-r}\right) + \frac{-r^{(n+1)}}{-r^{(n+1)}}\left(\frac{1 - \left(\frac{1}{r}\right)^{(n+1)}}{1 - \frac{1}{r}}\right)\\ &= \frac{r^n - r^{(2n+1)}}{r^n - r^{(n+1)}} + \frac{1 - r^{(n+1)}}{r^n - r^{(n+1)}} = \frac{r^n\left(1 - r^{(n+1)}\right) + \left(1 - r^{(n+1)}\right)}{r^n - r^{(n+1)}}\\ &= \frac{r^n(2) + 2}{r^n + 1} = \frac{2(r^n + 1)}{(r^n + 1)} = 2\end{aligned}$$

And finally, putting it all together we get

$$\sum_{k=1}^{n}\left(2 - 2\cos\left(\frac{k\pi}{n+1}\right)\right) = 2n + \sum_{k=0}^{n}\left(\exp\left(\frac{ik\pi}{n+1}\right) + \exp\left(\frac{-ik\pi}{n+1}\right)\right) - 2 = 2n + 2 - 2 = 2n$$

9) We now show that $K_3 = {\Delta_-}^T\Delta_-$ and $B_4 = \Delta_-{\Delta_-}^T$, where Δ_- is the 4×3 backward difference matrix given by

$$\Delta_- = \begin{bmatrix} 1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1 \end{bmatrix}$$

$$K_3 = {\Delta_-}^T\Delta_- = \begin{bmatrix} 1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2 \end{bmatrix}$$

$$B_4 = \Delta_-{\Delta_-}^T = \begin{bmatrix} 1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1 \end{bmatrix}$$

13) It is known that if matrices A and B commute, $AB = BA$, then they have the same eigenvectors and the eigenvalue matrix of $(A + B)$ is the sum of the eigenvalues of A and B. Now in this case, $B = 2I$, and it certainly commutes with A. The eigenvalue matrix of $2I$ is simply $2I$. And if $A = S\Lambda S^{-1}$, meaning it has the eigenvalue matrix Λ, then the eigenvalue matrix of $(A + 2I)$ is $(\Lambda + 2I)$. Also, the eigenvector matrix of $(A + 2I)$ is the same as the eigenvector matrix of A; it is simply S. As a check we can show that $(A + 2I) = S(\Lambda + 2I)S^{-1}$:

$$A+2I = SS^{-1}(A+2I)SS^{-1} = S(S^{-1}AS+S^{-1}(2I)S)S^{-1} = S(S^{-1}(S\Lambda S^{-1})S+2S^{-1}S)S^{-1} = S(\Lambda+2I)S^{-1}$$

1.6: 2, 8

2) $u^T K u = 4u_1^2 + 16u_1u_2 + 26u_2^2 = [(2u_1)^2 + 2(2u_1)(4u_2) + (4u_2)^2] + 10u_2^2 = (2u_1 + 4u_2)^2 + 10u_2^2$

$$K = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} = LDL^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Now the $\texttt{chol}(K) = \sqrt{D}L^T$ can be calculated because L and D are known. Also, the square root of a diagonal matrix, D, is simply a diagonal matrix consisting of the square roots of the entries on the diagonal, or

$$\sqrt{D} = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & \sqrt{10} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{10} \end{bmatrix}$$

So, we finally calculate $\texttt{chol}(K)$ and it is

$$\texttt{chol}(K) = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{10} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & \sqrt{10} \end{bmatrix}$$

Finding the Cholesky factorization in MATLAB using the `chol` command gives the same answer.

8) The follow four 2×2 matrices are tested to see if they are positive definite and thus have two positive eigenvalues. If the matrix has the following structure

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

then the two tests that must be satisfied for the matrix A to be positive definite are: i) $a > 0$ and ii) $ac > b^2$.

$A_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}$	$A_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}$	$A_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}$	$A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}$
$a = 5 > 0$	$a = -1 \ngtr 0$	$a = 1 > 0$	$a = 1 > 0$
$ac = 35 \ngtr b^2 = 36$	$ac = 5 > b^2 = 4$	$ac = 100 \ngtr b^2 = 100$	$ac = 101 > b^2 = 100$
Does not have two positive eigenvalues	Does not have two positive eigenvalues	Does not have two positive eigenvalues	Does have two positive eigenvalues

We are now interested in finding a vector u such that $u^T A_1 u < 0$. One example of such a vector is the column vector (-1.2, 1).

$$u^T A_1 u = \begin{bmatrix} -1.2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} -1.2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.2 \end{bmatrix} = -0.2 < 0$$