Problem 1

Let $\Omega \subset \mathbb{R}^2$ denote a simply connected domain with boundary $\partial \Omega$, and let $F = u(x, y) \mathbf{i} + v(x, y) \mathbf{j}$. The divergence theorem in two dimensions is stated as:

$$\int_{\Omega} \nabla \cdot F \, dA = \int_{\partial \Omega} F \cdot \mathbf{n} \, ds. \quad (1)$$

We will show that this is true for the special case of a rectangle. Suppose $\Omega$ is given by the rectangle $[a, b] \times [c, d]$, where $b > a$ and $d > c$. Let:

- $\partial \Omega_1$ = bottom edge of the rectangle
- $\partial \Omega_2$ = right edge of the rectangle
- $\partial \Omega_3$ = top edge of the rectangle
- $\partial \Omega_4$ = left edge of the rectangle

Hence, $\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2 \cup \partial \Omega_3 \cup \partial \Omega_4$. Then, Equation (1) becomes

$$\int_{\Omega} \nabla \cdot F \, dA = \int_{\Omega} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \, dA$$

$$= \int_{a}^{b} \int_{c}^{d} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \, dx \, dy$$

$$= \int_{a}^{b} \int_{c}^{d} \frac{\partial u}{\partial x} \, dx \, dy + \int_{c}^{d} \int_{a}^{b} \frac{\partial v}{\partial y} \, dy \, dx$$

$$= \left[ u(y, c) - u(y, a) \right] \, dy + \int_{a}^{b} \left[ v(d, x) - v(c, x) \right] \, dx$$

$$= \int_{c}^{d} u(b, y) \, dy - \int_{c}^{d} u(a, y) \, dy + \int_{a}^{b} v(x, d) \, dx - \int_{a}^{b} v(x, c) \, dx$$

$$= -\int_{a}^{b} v(x, c) \, dx + \int_{c}^{d} u(b, y) \, dy + \int_{a}^{b} v(x, d) \, dx - \int_{c}^{d} u(a, y) \, dy$$

$$= \int_{\partial \Omega_1} F \cdot \mathbf{n} \, ds + \int_{\partial \Omega_2} F \cdot \mathbf{n} \, ds + \int_{\partial \Omega_3} F \cdot \mathbf{n} \, ds + \int_{\partial \Omega_4} F \cdot \mathbf{n} \, ds$$

So, we have shown $\int_{\Omega} \nabla \cdot F \, dA = \int_{\partial \Omega} F \cdot \mathbf{n} \, ds$ in the special case of the rectangle.
Problem 2a

Method 1:

Consider the function \( f(z) = \frac{1}{z(5-z)} \). First, we apply partial fraction decomposition to obtain:

\[
\begin{align*}
  f(z) &= \frac{1}{z(5-z)} = \frac{A}{z} + \frac{B}{5-z} \\
  \implies 1 &= A(5-z) + Bz \\
  \implies 1 &= 5A + (-A + B)z \\
  \implies 5A &= 1, \quad -A + B = 0
\end{align*}
\]

Thus, \( A = B = \frac{1}{5} \) and

\[
f(z) = \frac{1}{5z} + \frac{1}{5(5-z)} = \frac{1}{5} \left( \frac{1}{z} - \frac{1}{z-5} \right)
\]

In this particular problem, \( f \) has singularities at \( z_1 = 0 \) and \( z_2 = 5 \). Since the Taylor series is to be expanded about the point \( z_0 = 1 \), it has a radius of convergence of \( R = 1 \) and converges throughout the disk \( |z-1| < 1 \).

We can simplify our work by employing the well-known series:

\[
\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots, \quad |z| < 1.
\]  

Expanded about the point \( z_0 = 1 \), this becomes

\[
\frac{1}{1-(z-1)} = \sum_{n=0}^{\infty} (z-1)^n = 1 + (z-1) + (z-1)^2 + \cdots, \quad |z-1| < 1.
\]  

Additionally, we use the other common series expansion:

\[
\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n, \quad |z| < 1.
\]

Expanded about the point \( z_0 = 1 \), this becomes

\[
\frac{1}{1+(z-1)} = \sum_{n=0}^{\infty} (-1)^n(z-1)^n, \quad |z-1| < 1.
\]

To take advantage of Equations (3) and (5), we write \( f \) as

\[
\begin{align*}
  f(z) &= \frac{1}{5} \left( \frac{1}{z} + \frac{1}{5-z} \right) \\
  &= \frac{1}{5} \left( \frac{1}{(z-1)+1} + \frac{1}{4-(z-1)} \right) \\
  &= \frac{1}{5} \left( \frac{1}{(z-1)+1} + \frac{1}{4} \cdot \frac{1}{1-(z-1)/4} \right)
\end{align*}
\]
Now, define $z - 1 = \frac{z-3}{4}$. Then,

$$f(z) = \frac{1}{5} \left[ \frac{1}{(z-1) + 1} + \frac{1}{4} \frac{1}{1 - (z-1)/4} \right]$$

$$= \frac{1}{5} \left[ \frac{1}{(z-1) + 1} + \frac{1}{4} \frac{1}{1 - (z-1)} \right]$$

$$= \frac{1}{5} \left[ \sum_{n=0}^{\infty} (-1)^n (z-1)^n + \frac{1}{4} \sum_{n=0}^{\infty} (z-1)^n \right], \quad |z-1| < 1$$

$$= \frac{1}{5} \left[ \sum_{n=0}^{\infty} (-1)^n (z-1)^n + \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z-1}{4}\right)^n \right], \quad |z-1| < 1$$

$$= \frac{1}{5} \left[ \sum_{n=0}^{\infty} (-1)^n (z-1)^n + \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z-1}{4}\right)^{n+1} \right], \quad |z-1| < 1$$

$$= \frac{1}{5} \sum_{n=0}^{\infty} \left[ (-1)^n + \frac{1}{4^{n+1}} \right] (z-1)^n, \quad |z-1| < 1$$

**Method 2:**

If a function, $f$, is analytic throughout the disk $|z - z_0| < R$ for some $z_0 \in \mathbb{C}$ and $R > 0$, then $f$ has the following Taylor Series representation:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$  \quad (6)

In this particular problem, $f$ has singularities at $z_1 = 0$ and $z_2 = 5$. Since the Taylor series is to be expanded about the point $z_0 = 1$, it has a radius of convergence of $R = 1$ because a circle with radius $R > 1$ would include the singularity of $f$ that occurs at $z = 0$. Consequently, the Taylor series for $f$ converges in the disk $|z-1| < 1$.

Now, we obtain a general expression for the $n$th derivative of $f$ evaluated at $z_0 = 1$.

$$f^{(n)}(1) = \frac{n!}{5} \left( (-1)^n + \frac{1}{4^{n+1}} \right)$$

Substituting this result for $f^{(n)}(1)$ into Equation (6), we obtain the Taylor series of $f$.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (z - 1)^n \quad |z-1| < 1$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left[ (-1)^n + \frac{1}{4^{n+1}} \right] (z - 1)^n \quad |z-1| < 1$$

$$= \frac{1}{5} \sum_{n=0}^{\infty} \left[ (-1)^n + \frac{1}{4^{n+1}} \right] (z - 1)^n \quad |z-1| < 1$$

Notice that the result here agrees with the final result from Method 1.
Problem 3

We seek to evaluate the integral

$$
\int_{-\infty}^{\infty} \frac{1}{(x^2 + 9)(x - 3i)(x - 5i)} \, dx = \int_{-\infty}^{\infty} \frac{1}{(x + 3i)(x - 3i)^2(x - 5i)} \, dx. \tag{7}
$$

Let \( f(x) = \frac{1}{(x + 3i)(x - 3i)^2(x - 5i)} \). We can write \( f(x) = \frac{p(x)}{q(x)} \), where \( p(x) = 1 \) and \( q(x) = (x + 3i)(x - 3i)^2(x - 5i) \). The zeros of \( q(x) \) are \( x_1 = -3i \), \( x_2 = 3i \), \( x_3 = 3i \), \( x_4 = 5i \). Of these roots, \( x_2 \), \( x_3 \), and \( x_4 \) lie above the real axis, and only \( x_1 \) lies below the real axis. For this reason, we will form a closed contour \( C_R \) that encloses the lower part of the half-plane. Now, we consider a transformation of the real integral in Equation (8) to an integral in the complex plane. Thus, we consider

$$
\int_{C_R} \frac{p(z)}{q(z)} \, dz
$$

The poles and closed contour \( C_R \) are shown below. Note that the pole at \( z = 3i \) has multiplicity two.

![Diagram showing the contour of integration](image)

Figure 1: Closed contour \( C_R \) of integration. Since only one pole is located below the half-plane, we choose the closed contour to be in the lower half-plane. Therefore, we only have to evaluate one residue.

Using the Cauchy Residue Theorem,

$$
\int_{C_R} \frac{p(z)}{q(z)} \, dz = -2\pi i \ \text{Res}(-3i) = -2\pi i \left( \frac{p(-3i)}{q'(-3i)} \right) = -2\pi i \left( \frac{1}{288i} \right) = -\frac{\pi}{144}.
$$

So, we conclude

$$
\int_{-\infty}^{\infty} \frac{1}{(x^2 + 9)(x - 3i)(x - 5i)} \, dx = -\frac{\pi}{144}. \tag{8}
$$