Problem 1

(1) - (1a) in p.24  Solve \( x(1 + x)y' + \frac{1 + x}{x} y = \sqrt{x} \).

**Solution** We try to divide both sides of the equation by \( x(1 + x) \). Before this step, we should test whether \( x = 0 \) and \( x = -1 \) are solutions to this equation.

With \( x = 0 \), we have:

\[
0 + \frac{1}{2} y = 0
\]  \hspace{1cm} (1)

This is only true for \( y = 0 \); for \( y > 0 \), (1) is not satisfied. So, \( x = 0 \) is not a solution to this equation.

With \( x = -1 \), we have:

\[
0 + 0 = i
\]  \hspace{1cm} (2)

Since we write \( \sqrt{x} \), that means we only consider the case \( x > 0 \).

Argument (2) is false, so we can safely divide both sides of the equation by \( x(1 + x) \), yielding:

\[
\frac{dy}{dx} + \frac{1}{2x} y = \frac{1}{\sqrt{x}(1 + x)}
\]  \hspace{1cm} (3)

whose integrating factor is:

\[
e^\int \frac{dx}{2x} = \sqrt{x}
\]  \hspace{1cm} (4)

Multiplying \( \sqrt{x} \) to both sides of (3):

\[
\sqrt{x} \frac{dy}{dx} + \frac{1}{2\sqrt{x}} y = \frac{1}{1 + x}
\]  \hspace{1cm} (5)

\((\sqrt{x}y)\)' = \frac{1}{1 + x}

\( \sqrt{x}y = C + \ln|x + 1| \)

So the solution is:

\[
y = \frac{C + \ln|x + 1|}{\sqrt{x}}
\]  \hspace{1cm} (6)

where \( C \) is an arbitrary constant.
If \(x, y\) are real numbers,

\[
y = \frac{C + \ln(x + 1)}{\sqrt{x}}. \tag{7}
\]

(2) - (1c) in p.25  Solve \(y' = \frac{1}{2} y + xy^2\).

**Solution**  We would like to make a change of variables such that \(u = \frac{1}{y}\). Before this step, we need to test whether \(y = 0\) is a solution to this equation.

With \(y = 0, y' = 0\). Therefore, for any \(x\) in the domain of \(y(x)\), we have:

\[
0 = 0 + 0 \tag{8}
\]

This is an identity, so we have found a singular point, \(y = 0\), of the equation.

When \(y \neq 0\), we conduct the change of variables such that \(u = \frac{1}{y}\):

\[
\frac{1}{u} = \frac{1}{ux} = \frac{1}{u^2} \quad \frac{1}{u^2} \quad \frac{1}{u^2} \quad \frac{1}{u^2} 
\]

Multiply the integrating factor, \(e^{\int \frac{1}{x} \, dx} = x\), to both sides:

\[
x \frac{du}{dx} + u = -x^2 
\]

\[
(ux)' = -x^2 \tag{10}
\]

\[
u = -\frac{1}{3} x^2 + \frac{c}{x},
\]

where \(c\) is an arbitrary constant.

Hence, the solution to the original equation is:

\[
y = 0 \tag{11}
\]

and

\[
y = \frac{3x}{C - x^2} \tag{12}
\]

where \(C\) is an arbitrary constant.

(3) - (2f) in p.25  Solve \(y'' + y' - y = xe^x + 7 \cosh x\).
Solution  First, we solve for the complementary solution $y_c$ that satisfies $y'' + y' - y = 0$. The polynomial of differential operator $D$ is:

$$p(D) = D^3 + D^2 - D - 1 = (D+1)(D^2-1) = (D+1)^2(D-1)$$  \hspace{1cm} (13)

So the associated homogeneous ODE $p(D)y = 0$ has solutions:

$$y_c = c_1 e^x + (c_2 + c_3 x) e^{-x},$$  \hspace{1cm} (14)

where $c_1$, $c_2$, and $c_3$ are arbitrary constants.

Next, we solve for a particular solution:

$$y_p = \frac{1}{(D+1)^2(D-1)}(x e^x + 7 \cosh x)$$

$$= \frac{1}{(D+1)^3(D-1)}[(x + \frac{7}{2})e^x + \frac{7}{2} e^{-x}]$$

$$= e^x \frac{1}{(D+2)^2} \frac{1}{(D-2)} \left( x + \frac{7}{2} \right) + \frac{7}{2} e^{-x} \frac{1}{D-2} e^{-x}$$

$$= e^x \frac{1}{(D+2)^2} \left( \frac{1}{2} x^2 + \frac{7}{2} x \right) + \frac{7}{2} e^{-x} \frac{1}{D-2} \frac{1}{2} x^2$$

$$= \frac{1}{8} e^x \left( 1 + \frac{1}{D} \right) (x^2 + 7x) - \frac{7}{8} e^{-x} \frac{1}{1 - \frac{1}{2} D} (x^2)$$

$$= \frac{1}{8} e^x \left[ 1 - (D + \frac{1}{4} D^2) + (D + \frac{1}{4} D^2)^2 + \cdots \right] (x^2 + 7x)$$

$$- \frac{7}{8} e^{-x} \left[ 1 + \frac{1}{2} D + \frac{1}{4} D^2 + \cdots \right] (x^2)$$

$$= \frac{1}{8} e^x (1 - D + \frac{3}{4} D^2)(x^2 + 7x) - \frac{7}{8} e^{-x} (1 + \frac{1}{2} D + \frac{1}{4} D^2)(x^2)$$

$$= \frac{1}{8} (x + 5x - \frac{11}{2}) e^x - \frac{7}{8} (x^2 + x + \frac{1}{2}) e^{-x},$$

Thus, the complete solution is:

$$y = (C_1 + \frac{1}{8} x^2 + \frac{5}{8} x) e^x + (C_2 + C_3 x - \frac{7}{8} x^2) e^{-x},$$  \hspace{1cm} (16)

where $C_1$, $C_2$, and $C_3$ are arbitrary constants.

(4)  Find the general solution of $(D + 1)^3 y = e^{-x}$.

Solution  First, we solve for the complementary solution $y_c$ that satisfies $(D + 1)^3 y = 0$. We assume $y_c = e^{-x} v(x)$, with the shift law we have:

$$(D + 1)^3 y = (D + 1)^3 [e^{-x} v(x)] = e^{-x} D^3 v(x) = 0.$$  \hspace{1cm} (17)
Since $e^{-x} \neq 0$,

\[ D^3 v(x) = 0, \]
\[ v(x) = c_1 + c_2 x + c_3 x^2, \]  
(18)

where $c_1$, $c_2$, and $c_3$ are arbitrary constants.

Hence,

\[ y_c = (c_1 + c_2 x + c_3 x^2)e^{-x}. \]  
(19)

where $c_1$, $c_2$, and $c_3$ are arbitrary constants.

Next, we find a particular solution to the original ODE. Divide both sides by $(D+1)^3$, we have:

\[ y_p = \frac{1}{(D+1)^3} e^{-x} \]
\[ = e^{-x} \frac{1}{D^3} e^{0} \]
\[ = e^{-x} \frac{1}{D^3}(1) \]
\[ = e^{-x} \left(-\frac{x^3}{6} + \frac{1}{2} c_4 x^2 + c_5 x + c_6\right), \]  
(20)

where $c_4$, $c_5$, and $c_6$ are arbitrary constants. For simplicity, we take $\frac{1}{D^3}(1) = \frac{1}{6} x^3$ so that

\[ y_p = \frac{1}{6} x^3 e^{-x}. \]  
(21)

This is because other particular solutions are just the sum of this particular solution and the complementary solution. Therefore, the complete solution to this ODE is:

\[ y = (C_1 + C_2 x + C_3 x^2 + \frac{1}{6} x^3)e^{-x}. \]  
(22)

where $C_1$, $C_2$, and $C_3$ are arbitrary constants.

**Problem 2**

(1) Find the real part and the imaginary part of $i^3$.

**Solution** Because

\[ i^3 = e^{i(\frac{\pi}{2} + 2\pi)(3i)} \]
\[ = e^{(-3)(\frac{\pi}{2} + 2\pi)} \]
\[ = e^{i(6n - \frac{3}{2})\pi} \]  
(23)

is a real number (with arbitrary integers $n = -k$),

the real part of $i^3$ is $e^{(6n - \frac{3}{2})\pi}$ ($n \in \mathbb{Z}$), and the imaginary part of $i^3$ is 0.
(2) Find the roots of \((z + 1)^4 = (z^2 - 1)^4\).

**Solution** Since \(z^2 - 1 = (z + 1)(z - 1)\), we have:

\[
(z + 1)^4 = (z^2 - 1)^4 = (z + 1)^4(z - 1)^4.
\]

(24)

When \((z + 1)^4 = 0\), we have a multiple root:

\[
z_1 = z_2 = z_3 = z_4 = 1.
\]

(25)

When \((z + 1)^4 \neq 0\), we can cancel this term, yielding:

\[
(z - 1)^4 = 1,
\]

(26)

with roots being:

\[
z = 1 + e^{\frac{2\pi i}{4}} = 1 + e^{\frac{\pi i}{2}},
\]

(27)

where \(n = 0, 1, 2, 3\), i.e.:

\[
z_5 = 2, z_6 = 1 + i, z_7 = 0, z_8 = 1 - i.
\]

(28)

Hence, the roots of \((z + 1)^4 = (z^2 - 1)^4\) are

\[
z = 0, 1, 2, 1 + i, 1 - i.
\]

(29)