Laureen Merouch

PSet 8

5/8/17

PDE's & Green's functions

① $\nabla^2 u = 0$

$0 < y < a$

$-\infty < x < \infty$

$u(x,0) = 0$

$u(x,a) = \dfrac{1}{1+x^2}$

Ⓘ $u(x,y)$ vanishes at $x \to \pm\infty$

By Ⓘ, we can take the fourier transform in x

$$u(x,y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} U(k,y) e^{ikx} \quad \text{where} \quad U(k,y) = \int_{-\infty}^{\infty} dx\, e^{-ikx} u(x,y)$$

$$\to \nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \int_{-\infty}^{\infty} \frac{dk}{2\pi} U(k,y) e^{ikx} = 0$$

$$D^2 \to -k^2$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \left(-k^2 + \frac{\partial^2}{\partial y^2}\right) U(k,y) = 0$$

$$\left(-k^2 + \frac{\partial^2}{\partial y^2}\right) U(k,y) = 0$$

$$D^2 - k^2 = 0 \to D = \pm k$$

$$U(k,y) = A(k) e^{ky} + B(k) e^{-ky}$$

B.C. $u(x,0) = 0 \to U(k,0) = 0$

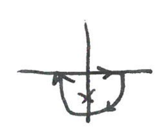
$$0 = A(k) + B(k) \to A(k) = -B(k) \to U(k,y) = A(k)\left[e^{ky} - e^{-ky}\right] = A(k) 2\sinh(ky)$$

B.C. $u(x,a) = \dfrac{1}{1+x^2}$

$$\frac{1}{1+x^2} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} A(k)\left(e^{ka} - e^{-ka}\right) \longrightarrow \tilde{f}(k) = \frac{1}{1+x^2} \;\&\; \tilde{f}(k) = A(k)\left(e^{ka} - e^{-ka}\right)$$

$$A(k)\left[e^{ka} - e^{-ka}\right] = \int_{-\infty}^{\infty} dx\, e^{-ikx} \frac{1}{1+x^2}$$: poles at $x = \pm i$ close contour for $k > 0$ & $k < 0$

$k > 0$: $\text{Res}(-i) = \dfrac{e^{-|k|}}{-2i}$



$k < 0$: $\text{Res}(i) = \dfrac{e^{-|k|}}{2i}$

$$A(k)\left(e^{ka} - e^{-ka}\right) = (-2\pi i)\frac{e^{-|k|}}{-2i} \;\&\; (2\pi i)\frac{e^{-|k|}}{2i} = \pi e^{-|k|}$$

$$A(k) = \frac{\pi e^{-|k|}}{e^{ka} - e^{-ka}} = \frac{\pi e^{-|k|}}{2\sinh(ka)}$$

$$u(k,y) = \frac{\pi e^{-|k|}}{2\sinh(ka)}\left[2\sinh(ky)\right]$$

$$u(x,y) = \int_{-\infty}^{\infty} \frac{dk}{2\cancel{\pi}} \cancel{\pi} e^{-|k|} \frac{\sinh(ky)}{\sinh(ka)} e^{ikx} = \underline{\int_{-\infty}^{\infty} dk\, e^{-|k|+ikx} \frac{1}{2} \frac{\sinh(ky)}{\sinh(ka)}}$$

② $$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2}, \quad \psi = \psi(x,t)$$

$$\psi(x,0) = f(x)$$

a. $$\tilde{\Psi}(k,t) = \int_{-\infty}^{\infty} dx\, e^{-ikx} \psi(x,t)$$

$$\left(i\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}\right)\psi = 0, \quad \psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \Psi(k,t) e^{ikx}$$

$$\left(i\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}\right)\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}\left(i\frac{\partial}{\partial t} - k^2\right)\Psi(k,t) = 0$$

$$\rightarrow \left(i\frac{\partial}{\partial t} - k^2\right)\Psi = 0$$

$$D = \frac{+k^2}{i} = -ik^2$$

$$\underline{\tilde{\Psi}(k,t) = A(k) e^{-ik^2 t}}$$

b. B.C. $\psi(x,0) = f(x)$

$$\left.\begin{array}{l} \rightarrow \Psi(k,0) = \tilde{f}(k) \\ \rightarrow \Psi(k,0) = A(k) \end{array}\right\} A(k) = \tilde{f}(k)$$

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k) e^{-ik^2 t + ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{i(kx - k^2 t)} \tilde{f}(k)$$

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{i(kx - k^2 t)} \int_{-\infty}^{\infty} dx'\, e^{-ikx'} f(x')$$

$$\underline{\psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx'\, f(x') e^{i(k(x-x') - k^2 t)}}$$

c. complete the square so that we can integrate w.r.t. k

$$i(k(x-x') - k^2 t) = -it\left(k^2 - \frac{k(x-x')}{t}\right) = -it\left[\left(k - \frac{(x-x')}{2t}\right)^2 - \frac{(x-x')^2}{4t^2}\right]$$

$$\rightarrow \psi(x,t) = \int_{-\infty}^{\infty} dx'\, f(x')\, e^{i\frac{(x-x')^2}{4t}} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-it\left[k - \frac{(x-x')}{2t}\right]^2}$$

let $k' = k - \frac{(x-x')}{2t}$

$dk' = dk$

$$\psi(x,t) = \int_{-\infty}^{\infty} dx'\, f(x')\, e^{i\frac{(x-x')^2}{4t}} \int_{-\infty}^{\infty} \frac{dk'}{2\pi} e^{-itk'^2}$$

let $u^2 = k'^2 t \rightarrow k'^2 = \frac{u^2}{t}$

$dk' = \frac{du}{\sqrt{t}}$

$$\psi(x,t) = \int_{-\infty}^{\infty} dx'\, f(x')\, e^{i\frac{(x-x')^2}{4t}} \int_{-\infty}^{\infty} \frac{du}{\sqrt{t}} \frac{1}{2\pi} e^{-iu^2}$$

$$\psi(x,t) = \int_{-\infty}^{\infty} dx'\, f(x')\, \underbrace{e^{i\frac{(x-x')^2}{4t}}\left(\frac{1}{2\pi}\sqrt{\frac{\pi}{t}}\, e^{-i\pi/4}\right)}_{G(x-x',t)}$$

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dx'}{2\pi} f(x')\, G(x-x',t),$$

③ Klein-Gordon Eq'n

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right)\phi(x,t) = 0$$

$-\infty < x < \infty$, $t > 0$, m a constant

$\phi(x,0) = f(x)$

$\phi_t(x,0) = g(x)$

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \Phi(k,t) e^{ikx}$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right)\phi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}\left(\frac{\partial^2}{\partial t^2} + k^2 + m^2\right)\Phi(k,t) = 0$$

$D^2 + k^2 + m^2 = 0 \rightarrow D = \pm i\sqrt{k^2+m^2}$

$$\Phi(k,t) = A(k)e^{i\sqrt{k^2+m^2}\,t} + B(k)e^{-i\sqrt{k^2+m^2}\,t}$$

B. C. $\phi(x,0) = f(x) \rightarrow \Phi(k,0) = \tilde{f}(k) \rightarrow \tilde{f}(k) = A(k) + B(k)$

B. C. $\phi_t(x,0) = g(x) \rightarrow \Phi_t(k,0) = A(k)\,i\sqrt{k^2+m^2} - B(k)\,i\sqrt{k^2+m^2} = g(k)$

solve system of equations

$$\tilde{f}(k) = A(k) + B(k)$$

$$+\quad g(k)\Big/\left(i\sqrt{k^2+m^2}\right) = A(k) - B(k)$$

$$\tilde{f}(k) + z\tilde{g}(k) = 2A(k) \quad \text{where} \quad z = \frac{1}{i\sqrt{k^2+m^2}}$$

$$\hookrightarrow A(k) = \frac{1}{2}\left(\tilde{f}(k) + z\tilde{g}(k)\right)$$

$$B(k) = \tilde{f}(k) - A(k) = \tilde{f}(k) - \frac{1}{2}\left(f(k) + z\tilde{g}(k)\right) = \frac{1}{2}\tilde{f}(k) - \frac{1}{2}z\tilde{g}(k)$$

$$\Phi(k,t) = \frac{1}{2}\left(\tilde{f}(k) + z\tilde{g}(k)\right)e^{z^{-1}t} + \frac{1}{2}\left(\tilde{f}(k) - z\tilde{g}(k)\right)e^{-z^{-1}t}$$

$$= \frac{1}{2}\left(\tilde{f}(k)e^{z^{-1}t} + \tilde{f}(k)e^{-z^{-1}t}\right) + \frac{z}{2}\left(\tilde{g}(k)e^{z^{-1}t} - \tilde{g}(k)e^{-z^{-1}t}\right)$$

$$\phi(x,t) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\left[\tilde{f}(k)\cos\left(\frac{z^{-1}}{i}t\right) + iz\tilde{g}(k)\sin\left(\frac{z^{-1}}{i}t\right)\right]$$

$$= \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\left(\tilde{f}(k)\cos\left(\sqrt{k^2+m^2}\,t\right) + \frac{\tilde{g}(k)}{\sqrt{k^2+m^2}}\sin\left(\sqrt{k^2+m^2}\,t\right)\right)$$

$$\phi(x,t) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\cos\sqrt{k^2+m^2}\,t\int_{-\infty}^{\infty}dx'\,e^{-ikx'}f(x') + \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\frac{\sin\sqrt{k^2+m^2}\,t}{\sqrt{k^2+m^2}}\int_{-\infty}^{\infty}dx'\,e^{-ikx'}g(x')$$

$$\phi(x,t) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}dx'\,e^{ik(x-x')}f(x')\cos\left(\sqrt{k^2+m^2}\,t\right) + \int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}dx'\,e^{ik(x-x')}g(x')\frac{\sin\left(\sqrt{k^2+m^2}\,t\right)}{\sqrt{k^2+m^2}}$$

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx' \, \underbrace{e^{ik(x-x')} \cos\left(\sqrt{k^2+m^2}\, t\right)}_{G_1(x-x',t)} f(x') + \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx' \, \underbrace{e^{ik(x-x')} \frac{\sin\left(\sqrt{k^2+m^2}\, t\right)}{\sqrt{k^2+m^2}}}_{G_2(x-x',t)} g(x')$$

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \left[G_1(x-x',t) f(x') + G_2(x-x',t) g(x') \right] dx'$$

G_1 and G_2 should probably include this 2π.

✓