Alex Sloboda — 18.075 - Pset 7

1. $\frac{d^2x}{dt^2} - 4x = \cos(t)$ $\quad t>0$

$x(0)=A,\ \dot{x}(0)=B$

a) $\mathcal{L}_L(s) = \int_0^\infty \left(\frac{d^2x}{dt^2} - 4x\right)e^{-st}\,dt = -x'(0) - sx(0) + s^2\mathcal{L}_x(s) - 4\mathcal{L}_x(s)$

$= -x'(0) - sx(0) + \mathcal{L}_x(s)(s^2-4) = -B - sA + \mathcal{L}_x(s)(s^2-4)$

$\mathcal{L}_R(s) = \int_0^\infty \cos(t)\,e^{-st}\,dt = \frac{s}{s^2+1} = -B - sA + \mathcal{L}_x(s)(s^2-4),\ \mathcal{L}_x(s) = \frac{\frac{s}{s^2+1} + B + sA}{s^2-4}$

$\mathcal{L}_x(s) = \frac{s}{(s^2+1)(s^2-4)} + \frac{B}{s^2-4} + \frac{sA}{s^2-4}$ $\quad \int_{3-i\infty}^{3+i\infty} \frac{s}{(s^2+1)(s^2-4)} e^{st} \frac{ds}{i2\pi} = -\frac{1}{5}\cos(t) + \frac{e^{2t}}{10} + \frac{e^{-2t}}{10}$

$\cancel{\frac{B}{2}\sin(2t) + A\cos(2t)}$

$\frac{1}{4}B(e^{2t} - e^{-2t}) + \frac{A}{2}(e^{2t} + e^{-2t}) - \frac{1}{5}\cos(t) + \frac{e^{2t}}{10} + \frac{e^{-2t}}{10}$

$= \boxed{\left(\frac{2A + B + \frac{2}{5}}{4}\right)e^{2t} + \left(\frac{2A - B + \frac{2}{5}}{4}\right)e^{-2t} - \frac{1}{5}\cos(t) = x(t)}$

b) $(D^2-4)x = \cos(t)$

$x_h = c_1 e^{-2t} + c_2 e^{2t}$

$(D^2-4)x_p = \cos(t),\ x_p = \frac{\cos(t)}{(D^2-4)} \Rightarrow \frac{\cos(t)}{-1-4} = \frac{\cos(t)}{-5}$

$\therefore x(t) = c_1 e^{-2t} + c_2 e^{2t} - \frac{\cos(t)}{5}$

✓

$x(0) = A = c_1 + c_2 - \frac{1}{5}$

$x'(0) = B = -2c_1 + 2c_2 + \frac{\sin(0)}{5} = -2c_1 + 2c_2$

$c_1 = A - c_2 + \frac{1}{5},\ B = 2c_2 - 2(A - c_2 + \frac{1}{5}) = 4c_2 - 2A - \frac{2}{5},\ c_2 = \frac{B + 2A + \frac{2}{5}}{4}$

$c_1 = \frac{4A - B - 2A - \frac{2}{5} + \frac{4}{5}}{4} = \frac{2A - B + \frac{2}{5}}{4} \therefore \boxed{x(t) = \frac{2A - B + \frac{2}{5}}{4}e^{-2t} + \frac{B + 2A + \frac{2}{5}}{4}e^{2t} - \frac{\cos(t)}{5}}$

c) I find the second method easier though less efficient since it requires more steps (find x_h, x_p, plug in initial conditions,..)

2. $x = \int_0^x \frac{1}{\sqrt{x-x'}} y(x')dx' \quad x>0$

$x = \frac{1}{\sqrt{x}} * y(x), \quad \mathcal{L}_x(s) = \mathcal{L}_{\frac{1}{\sqrt{x}}}(s) \cdot \mathcal{L}_y(s)$

$\frac{1}{s^2} = \mathcal{L}_{\frac{1}{\sqrt{x}}}(s)\, \mathcal{L}_y(s), \quad \mathcal{L}_{\frac{1}{\sqrt{x}}}(s) = \int_0^\infty \frac{1}{\sqrt{x}} e^{-sx} dx = \frac{\sqrt{\pi}}{\sqrt{s}}$

$\therefore \mathcal{L}_y(s) = \frac{\sqrt{s}}{\sqrt{\pi} s^2} = \frac{\sqrt{\pi}}{\pi s \sqrt{s}} \quad \therefore y(x) = \frac{2\sqrt{x}}{\pi}$ ✓