Problem Set 6

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18.075
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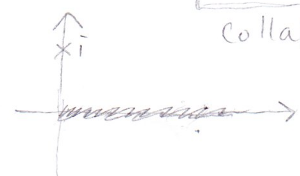
Problem 1.

11a) Evaluate, making use of branch cuts:



$$I \equiv \int_0^\infty \frac{\ln^2 x}{1+x^2}\,dx \qquad (\text{ans. } \pi^3/8)$$

$$z = re^{i\theta} \qquad \ln^2 z = \ln^2 r + i\theta \ln r - \theta^2$$

$$\ln z = \ln r + i\theta \qquad z^2 = r^2 e^{2i\theta}$$

$$J \equiv \oint \frac{\ln^2 z}{1+z^2}\,dz = \int_{-\infty}^{-\varepsilon} \frac{\ln^2 z}{1+z^2}dz + \int_{-\varepsilon}^{\varepsilon} \frac{\ln^2 z}{1+z^2}dz + \int_{\varepsilon}^{\infty} \frac{\ln^2 z}{1+z^2}dz + \cancel{\int \frac{\ln^2 z}{1+z^2}dz} \to 0$$

$$= \int_0^\infty \frac{\ln^2 r + i\pi \ln r - \pi^2}{1+r^2}\,dr + \cancel{\int_{-\varepsilon}^{\varepsilon} \frac{\ln^2 \varepsilon + i\theta \ln \varepsilon - \theta^2}{1+\varepsilon^2} \varepsilon i e^{i\theta}\,d\theta} + \int_0^\infty \frac{\ln^2 r}{1+r^2}\,dr$$

$1+\varepsilon^2 \approx 1$

$$= 2\int_0^\infty \frac{\ln^2 r}{1+r^2}\,dr + \underbrace{\int_0^\infty \frac{i\pi \ln r}{1+r^2}\,dr}_{0} + \int_0^\infty \frac{-\pi^2}{1+r^2}\,dr$$

$$J \equiv \oint \frac{\ln^2 z}{1+z^2}\,dz = 2\pi i R \qquad R = \lim_{z\to i} \frac{(z-i)}{(z-i)(z+i)} \ln^2 z = \frac{\ln^2 i}{2i} = \frac{\cancel{\ln^2 1} + \cancel{i\frac{\pi}{2}\ln 1} - \left(\frac{\pi}{2}\right)^2}{2i}$$

$$J = \frac{-2\pi^3}{2} = -\frac{\pi^3}{4}$$

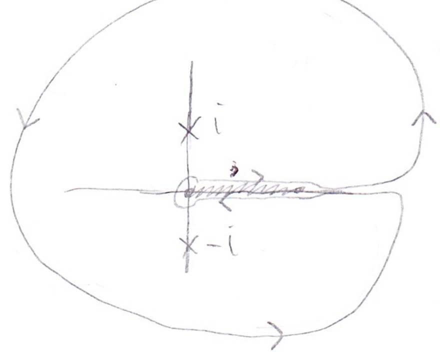
$$\int_0^\infty \frac{i\pi \ln r}{1+r^2}\,dr = 0 \quad \text{from lecture 18}$$

$$-\pi^2 \int_0^\infty \frac{1}{1+r^2}\,dr \to (-\pi^2)\tan^{-1} r \Big|_0^\infty = \left(\frac{\pi}{2} - 0\right)(-\pi^2) = \frac{-\pi^3}{2}$$

$$J = -\frac{\pi^3}{4} = 2I - \frac{\pi^3}{2} \qquad I = \frac{1}{2}\left(-\frac{\pi^3}{4} + \frac{2\pi^3}{4}\right) = \boxed{\frac{\pi^3}{8}}$$

✓

Problem 2



$$I = \int_0^1 \left(\frac{x}{1-x}\right)^{1/3} \frac{1}{1+x^2}\,dx$$

$z^{1/3}$ has branch point at 0

$(1-z)^{1/3}$ has branch point at 1

let $z = \frac{1}{r}$ $\quad \left(\frac{z}{1-z}\right)^{1/3} = \left(\frac{1/r}{1-1/r}\right)^{1/3} = \left(\frac{1}{r-1}\right)^{1/3}$

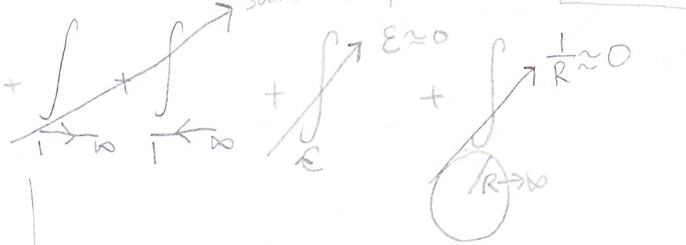
no branch point at $r=0$

∴ no branch point at $z=\infty$

$$J \equiv \oint_C \left(\frac{z}{1-z}\right)^{1/3} \frac{1}{1+z^2}\,dz$$

same value, cancels

$$J = \int_{0\to 1} \left(\frac{z}{1-z}\right)^{1/3} \frac{1}{1+z^2}\,dz + \int_{0\leftarrow 1} \left(\frac{z}{1-z}\right)^{1/3} \frac{1}{1+z^2}\,dz + \int_{1\to\infty} + \int_{1\leftarrow\infty} + \int_{\varepsilon} + \int_{R}$$



$z = re^{i\theta}$

$$\left(\frac{z}{1-z}\right)^{1/3} \begin{cases} \theta = 0 & \left(\frac{r}{1-r}\right)^{1/3} \\ \theta = 2\pi & \left(\frac{r^{1/3} e^{2\pi i/3}}{(1-r)^{1/3}}\right) \end{cases}$$

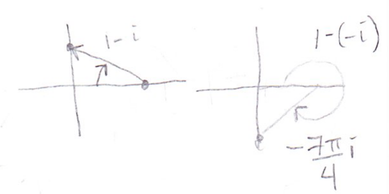
$$J = \int_0^1 \left(\frac{x}{1-x}\right)^{1/3} \frac{1}{1+x^2}\,dx + \int_1^0 \left(\frac{x}{1-x}\right)^{1/3} \frac{1}{1+x^2} e^{2\pi i/3}\,dx = I - e^{2\pi i/3} I$$

$J = 2\pi i R$

$$R_1 = \left(\frac{z}{1-z}\right)^{1/3} \frac{1}{2i}\Bigg|_{z=i} = \left(\frac{i}{1-i}\right)^{1/3} \left(\frac{1}{2i}\right)$$

$$R_2 = \left(\frac{z}{1-z}\right)^{1/3}\Bigg|_{z=-i} \left(\frac{1}{-2i}\right) = \left(\frac{-i}{1+i}\right)^{1/3} \left(\frac{-1}{2i}\right)$$

$$J = \pi i \left(\left(\frac{e^{i\pi/2}}{\sqrt{2}\, e^{-i\pi/4}}\right)^{1/3} (-i) + \left(\frac{e^{3\pi i/2}}{\sqrt{2}\, e^{-7i\pi/4}}\right)^{1/3} (i) \right)$$



the arguments are specified b/c of branch cut

$$J = \pi \left(\frac{e^{i\pi/12}}{\sqrt{2}} - \frac{e^{-i\pi/12}}{\sqrt{2}} \right)$$

$$I = \frac{\pi}{\sqrt{2}} \left(e^{i\pi/12} - e^{-i\pi/12}\right) \left(\frac{1}{1-e^{2\pi i/3}}\right)$$

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$$I = \frac{-(\sqrt{3}-3)\pi}{3 \cdot 2^{2/3}}$$

Problem 3

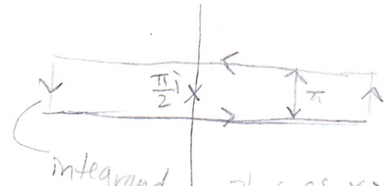
$$f(x) = \frac{1}{\cosh x} \qquad \cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$\tilde{f}(k) \equiv \int_{-\infty}^{\infty} \frac{e^{-ixk}}{\cosh x}\,dx = \int_{-\infty}^{\infty} \frac{2e^{-ixk}}{e^{x}+e^{-x}}\,dx \equiv I$$

$$\cosh(z+i\pi) = e^{z}e^{i\pi} + e^{-z}e^{-i\pi} = -e^{z}-e^{-z} = -\cosh(z)$$

$$\cosh(x=+\infty) = e^{\infty}+e^{-\infty} = e^{\infty} \qquad \cosh(x=-\infty) = e^{-\infty}+e^{\infty} = e^{\infty}$$

$$e^{-izk} = e^{-ixk},\ z = x \qquad e^{-ik(x+\pi i)} = e^{-ikx}e^{\pi k}$$

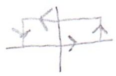


integrand vanishes as $x \to \pm\infty$.

$$e^{z}+e^{-z} = 0 \qquad e^{z} = -e^{-z} \qquad e^{2z} = -1 = e^{i\pi}e^{2in\pi}$$

$$z = \frac{i\pi}{2} + \frac{in\pi}{2} \quad \text{poles of integrand}$$

$$J \equiv \oint \frac{2e^{-izk}}{e^{z}+e^{-z}}\,dz = \int_{-\infty}^{\infty} \frac{e^{-ixk}}{\cosh x}\,dx + \int_{\infty}^{-\infty} \frac{e^{-ixk}e^{\pi k}}{-\cosh x} = I + e^{\pi k} I$$



simple pole at $\frac{\pi}{2}i$

$$J = 2i\pi R \qquad R = \lim_{z\to\frac{\pi i}{2}} \left(z-\frac{\pi i}{2}\right)\frac{e^{-izk}}{\cosh z} = \lim_{z\to\frac{\pi i}{2}} \frac{e^{-izk} - ikze^{-izk} - \frac{\pi}{2}ke^{-izk}}{\sinh z}$$

$$R = \frac{e^{+\pi k/2} + \cancel{ik\left(\frac{\pi i}{2}\right)e^{+\pi k/2}} \cancel{-\frac{\pi}{2}ke^{+\pi k/2}}}{\sinh\frac{\pi i}{2}}$$

$$I = \frac{2i\pi}{\sinh\left(\frac{\pi i}{2}\right)}\,\frac{e^{\pi k/2}}{1+e^{\pi k}}$$

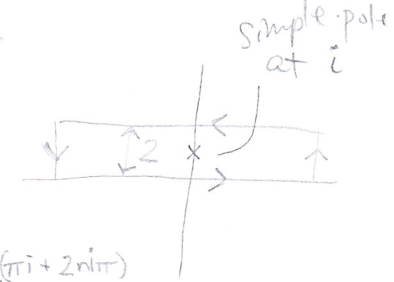
$$I = \frac{4i\pi e^{\pi k/2}}{\left(e^{\pi i/2}-e^{-\pi i/2}\right)\left(1+e^{\pi k}\right)} = \frac{4i\pi e^{\pi k/2}}{(2i)(1+e^{\pi k})} = \frac{2\pi e^{\pi k/2}}{1+e^{\pi k}}$$

$$\tilde{f}(k) = \frac{2\pi e^{\pi k/2}}{1+e^{\pi k}}\left(\frac{e^{-\pi k/2}}{e^{-\pi k/2}}\right) = \frac{2\pi}{e^{-\pi k/2}+e^{\pi k/2}} = \boxed{\frac{\cancel{2}\pi}{\cosh\frac{\pi k}{2}} = \tilde{f}(k)}$$

Problem 3, cont'd

Inversion $\quad f(x) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \frac{dk}{2\pi}$

$$f(x) = \int_{-\infty}^{\infty} \frac{\cancel{2\pi}}{\cosh \frac{\pi k}{2}} e^{ikx} \frac{dk}{\cancel{2\pi}} = \int_{-\infty}^{\infty} \frac{e^{ikx}}{\cosh \frac{\pi k}{2}} dk \equiv I$$



$$\cosh\left(\tfrac{\pi}{2}(k+2i)\right) = e^{\pi k/2} e^{\pi i} + e^{-\pi k/2} e^{-\pi i} = -\cosh \tfrac{\pi k}{2}$$

$$e^{\pi k/2} + e^{-\pi k/2} = 0 \qquad e^{\pi k/2} e^{+\pi k/2} = -1 \qquad e^{\pi k} = e^{(\pi i + 2n i \pi)}$$

$$k = i + 2ni$$

$$J \equiv \oint \frac{e^{ikx}}{\cosh \frac{\pi k}{2}} dk = \int_{-\infty}^{\infty} \frac{e^{ikx}}{\cosh \frac{\pi k}{2}} + \int_{\infty}^{-\infty} \frac{e^{ix(k+2i)}}{-\cosh \frac{\pi k}{2}} = I + e^{-2x} I$$

$$J = 2\pi i R \qquad R = \lim_{k \to i} (k-i) \frac{e^{ikx}}{\cosh \frac{\pi k}{2}} = \lim_{k \to i} \frac{kixe^{ikx} + e^{ikx} + xe^{ikx}}{\left(\frac{\pi}{2}\right) \sinh \frac{\pi k}{2}} = \frac{2e^{-x}}{\pi \sinh \frac{\pi i}{2}}$$

$$I = \frac{4ie^{-x}}{e^{\pi i/2} - e^{-\pi i/2}} \frac{1}{1+e^{-2x}} = \frac{2e^{-x}}{1+e^{-2x}} \left(\frac{e^{x}}{e^{x}}\right) = \frac{2}{e^{x}+e^{-x}} = \frac{1}{\cosh x}$$

$f(x) = \frac{1}{\cosh x}$ the inversion formula is valid

Problem 4

$f(\theta)=\theta, \quad 0<\theta<\pi$

cos is even $\rightarrow f(\theta) = -\theta \quad -\pi<\theta<0$

a) $f(\theta) = a_0 + \sum_{n=1}^{\infty} A_n \cos n\theta$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\left(2\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$A_n = a_n + a_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\,d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta} f(\theta)\,d\theta$$

$$= \frac{-1}{2\pi}\int_{-\pi}^{0} e^{-in\theta}\theta\,d\theta + \frac{-1}{2\pi}\int_{-\pi}^{0} e^{in\theta}\theta\,d\theta + \frac{1}{2\pi}\int_{0}^{\pi} e^{-in\theta}\theta\,d\theta + \frac{1}{2\pi}\int_{0}^{\pi} e^{in\theta}\theta\,d\theta$$

integrate by parts, let $u=\theta$, $dv = e^{-in\theta}d\theta$

$du = d\theta$, $v = \frac{-1}{in}e^{-in\theta}$

$$\int e^{-in\theta}\theta\,d\theta = \frac{-\theta}{in}e^{-in\theta} - \int \frac{-1}{in}e^{-in\theta}\,d\theta = \frac{-\theta}{in}e^{-in\theta} + \frac{1}{n^2}e^{-in\theta}$$

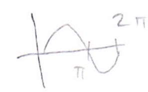
$$\int e^{in\theta}\theta\,d\theta = \frac{\theta}{in}e^{in\theta} + \frac{1}{n^2}e^{in\theta}$$

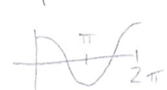
$$A_n = \frac{-1}{2\pi}\left[\frac{-\theta}{in}e^{-in\theta} + \frac{1}{n^2}e^{-in\theta} + \frac{\theta}{in}e^{in\theta} + \frac{1}{n^2}e^{in\theta}\right]\Bigg|_{-\pi}^{0}$$

$$+ \frac{1}{2\pi}\left[\frac{-\theta}{in}e^{-in\theta} + \frac{1}{n^2}e^{-in\theta} + \frac{\theta}{in}e^{in\theta} + \frac{1}{n^2}e^{in\theta}\right]\Bigg|_{0}^{\pi}$$

$$= \frac{\pm 1}{2\pi}\left[\frac{-2}{n^2} + \left(\frac{\pi}{in}e^{in\pi} + \frac{e^{in\pi}}{n^2} + \frac{-\pi}{in}e^{-in\pi} + \frac{1}{n^2}e^{-in\pi}\right)\right]$$

$$+ \frac{1}{2\pi}\left[\frac{-\pi}{in}e^{-in\pi} + \frac{1}{n^2}e^{-in\pi} + \frac{\pi}{in}e^{in\pi} + \frac{1}{n^2}e^{in\pi} - \frac{2}{n^2}\right]$$





$$= \frac{1}{2\pi}\left(-\frac{4}{n^2} + \frac{4\pi}{n}\cancel{\sin n\pi} + \frac{4}{n^2}\cos n\pi\right)$$

$\sin n\pi = 0 \ \forall n$ integer

$$= \frac{-2}{\pi n^2}(1-\cos n\pi)$$

$\cos n\pi = 1 \quad n$ even

$= -1 \quad n$ odd

$A_n = 0 \quad n$ even

$= \frac{-4}{\pi n^2} \quad n$ odd

$$f(\theta) = \frac{\pi}{2} + \sum_{n=1}^{\infty} A_n \cos n\theta, \quad A_n = 0 \ n \text{ even}, \ = \frac{-4}{\pi n^2} \ n \text{ odd}$$

b) $f(\theta) = \frac{\pi}{2} + \sum_{n=1}^{\infty} A_n \cos n\theta$, $A_n = 0$ n even

$= \frac{-4}{\pi n^2}$ n odd

set $\theta = 0$, $\cos n\theta = 1$

$f(0) = 0$

$0 = \frac{\pi}{2} + \frac{-4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$

$\frac{\pi}{2} \cdot \frac{\pi}{4} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

c) $\frac{1}{\pi} \int_0^{\pi} f^2(\theta)\, d\theta = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2$

$\frac{1}{\pi} \int_0^{\pi} \theta^2\, d\theta = \frac{1}{\pi} \frac{\theta^3}{3}\Big|_0^{\pi} = \frac{\pi^2}{3}$

$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{1}{2}\left(\frac{16}{\pi^2}\right) \sum_{n=1}^{\infty} \frac{1}{n^4}$ for n odd.

$\frac{\pi^2}{3} - \frac{\pi^2}{4} = \frac{\pi^2}{12} = \frac{8}{\pi^2}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right)$

$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$

c) converges faster because the latter terms are more neglagible since it is $\frac{1}{n^4}$ as opposed to $\frac{1}{n^2}$ as in the series in b). ✓