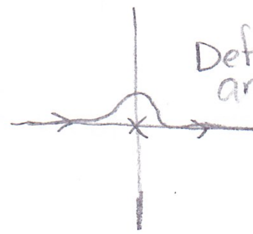
3/24/17 18.075 PSET 3 Ben Harpt

Collaborators: Yukino Nagai

1. a) $$I = \int_{-\infty}^{\infty} \frac{1-\cos(3x)}{x^2}\,dx$$



Deform contour around singularity at $z=0$

$$= \int_{-\infty}^{\infty} \frac{1-\frac{1}{2}(e^{i3x}+e^{-i3x})}{x^2}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{2-e^{i3x}-e^{-i3x}}{x^2}\,dx$$

$$= \frac{1}{2}\cancel{\int \frac{2-e^{i3z}}{z^2}\,dz}^{\to 0} - \frac{1}{2}\int \frac{-e^{-i3z}}{z^2}\,dz$$

$$= \frac{1}{2}\cdot 2\pi i \, Res(0)$$

$$= \pi i(-3i)$$

$$= 3\pi$$

$$Res(0) = \frac{d}{dz}\left(z^2 f(z)\right)\Big|_{z=0}$$
$$= \frac{d}{dz}\left(e^{-i3z}\right)\Big|_{z=z_0}$$
$$= -3i\,e^{-i3z}\Big|_{z=z_0}$$
$$= -3i$$

b) $$I \equiv P\int_{-\infty}^{\infty} \frac{1-\cos(3x)}{x^2}\,dx$$ Now define $$J \equiv P\int_{-\infty}^{\infty} \frac{1-e^{i3x}}{x^2}\,dx$$

$$\Rightarrow I = Re(J)$$

$$K \equiv \underbrace{\int \frac{1-e^{i3z}}{z^2}\,dz}_{=0 \text{ when we close the contour upstairs}} = J + \int \frac{1-e^{i3z}}{z^2}\,dz = 0$$

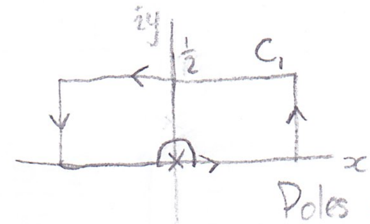
$$\Rightarrow \quad J = -\int_{\frown} \frac{1-e^{i3z}}{z^2}\,dz = -\int_{\frown} \frac{\cancel{1}-(\cancel{1}+i3z-\cancel{\frac{9}{2}z^2}^{\,\to 0}-\ldots)}{z^2}\,dz$$

$$= 3i\int_{\frown} \frac{1}{z}\,dz$$

✓ Now $\oint_{\odot} \frac{dz}{z} = 2\pi i \Rightarrow \int_{\frown} \frac{dz}{z} = -\pi i \Rightarrow J = 3\pi$

$$I = \mathrm{Re}(J) = 3\pi$$

2. $I = \int_{-\infty}^{\infty} \frac{x}{\sinh(\pi x)}\,dx$



Poles at $z = ni$

$$\sinh(\pi z) = \frac{e^{\pi x}e^{i\pi y} - e^{-\pi x}e^{-i\pi y}}{2}$$

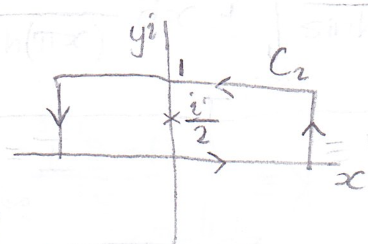
$\underline{z \to z+\frac{i}{2}}$) $\quad \sinh\left(\pi z + \frac{\pi}{2}i\right) = \frac{e^{\pi x}e^{i\pi y}e^{i\pi/2} - e^{-\pi x}e^{-i\pi y}e^{-i\pi/2}}{2} = i\cosh(\pi z)$

$$J \equiv \oint_{C_1} \frac{z}{\cosh(\pi z)}\,dz = I - \int_{-\infty}^{\infty} \frac{x+i/2}{i\cosh(\pi x)}\,dx = 0$$

(Since our contour encloses no singularity)

$$\Rightarrow I + i\int_{-\infty}^{\infty} \cancel{\frac{x}{\cosh(\pi x)}}^{\,\to 0}\,dx - \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\cosh(\pi x)}\,dx = 0$$

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\cosh(\pi x)}\,dx$$



$\underline{z \to z+i}$)

$$K \equiv \oint_{C_2} \frac{1}{\cosh(\pi z)}\,dz = 2I - \underbrace{\int_{-\infty}^{\infty} \frac{1}{-\cosh(\pi x)}\,dx}_{=-2I} = 4I$$

$$\Rightarrow I = \frac{1}{4}K = \frac{1}{4}\left(2\pi i \cdot \mathrm{Res}\left(\frac{i}{2}\right)\right)$$
$$= \frac{1}{4} \cdot 2\pi i \cdot \frac{1}{\pi i}$$
$$= \frac{1}{2}$$

$$\mathrm{Res}\left(\frac{i}{2}\right) = \left.\frac{1}{\pi \sinh(\pi z)}\right|_{z=\frac{i}{2}}$$
$$= \frac{2}{\pi} \left.\frac{1}{e^{\pi z} - e^{-\pi z}}\right|_{z=\frac{i}{2}}$$
$$= \frac{2}{\pi} \frac{1}{i-(-i)}$$
$$= \frac{1}{\pi i}$$

3. $f(z) = (z^3-1)^{1/3}$

$f(z) = 0$ when $z = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$\therefore$ These are branch points because the exponent $\frac{1}{3}$ on $f(z)$ is not an integer.

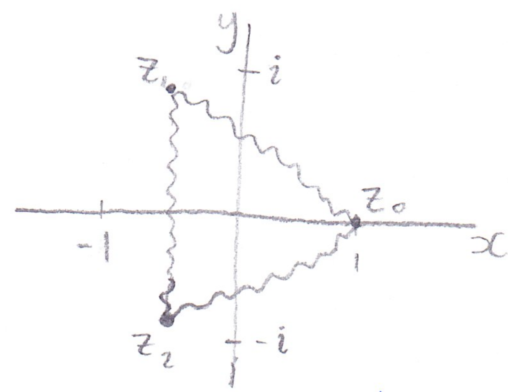
$$z^3 - 1 = 0$$
$$z^3 = 1 = e^{i2\pi n}$$
$$z = e^{i2\pi n/3}$$
$$\Rightarrow z_0 = e^0 = 1$$
$$z_1 = e^{i2\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$z_2 = e^{i4\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Is $\infty$ a branch point?

Let $z = \frac{1}{\omega}$ $\Rightarrow f(\omega) = \left(\frac{1}{\omega^3} - 1\right)^{1/3}$
$$= \left[\frac{1}{\omega^3}(1-\omega^3)\right]^{1/3}$$
$$= \frac{1}{\omega}(1-\omega^3)^{1/3}$$

$\omega = 0$ is not a branch point since the exponent 1 on $\frac{1}{\omega}$ is an integer.

$\therefore$ $z = \infty$ is NOT a branch point




Would also work with just two cuts.