18.0751
PSet 3
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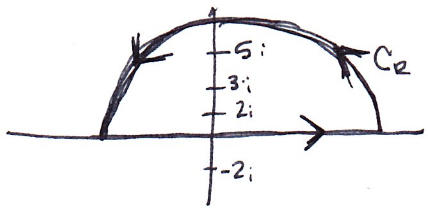
1a.

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)(x-3i)(x-5i)} = \int_{-\infty}^{\infty} \frac{dx}{(x+2i)(x-2i)(x-3i)(x-5i)}$$

poles at $2i, -2i, 3i, 5i$



$\lim x \to 2i = \frac{1}{(4i)(-i)(-3i)} = \frac{1}{-12i}$

$\lim x \to 3i = \frac{1}{10i}$

$\lim x \to 5i = \frac{1}{-42i}$

$$\boxed{I = 2\pi i\left[\frac{-1}{12i} + \frac{1}{10i} - \frac{1}{42i}\right] = \frac{-\pi}{70}}$$

✓

1b. $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

Call $z = e^{i\theta}$, then $dz = iz\,d\theta$ and $\cos\theta \to \frac{z+z^{-1}}{2}$

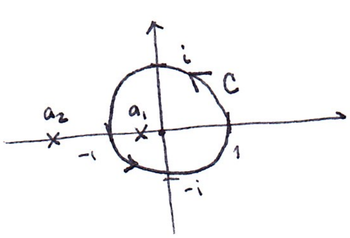
$$I = \int \frac{dz}{iz\left(2+\frac{1}{2}(z+z^{-1})\right)^2} = \int \frac{dz}{iz\left[\left(\frac{1}{2}\right)^2\left(4+(z+z^{-1})\right)^2\right]} = \int \frac{dz}{iz\left[\left(\frac{1}{2z}\right)^2\left(4z+z^2+1\right)^2\right]}$$

$$= \int \frac{4z\,dz}{i\left(z^2+4z+1\right)^2}$$

solve for poles using quadratic formula: $z = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$

two poles: $a_1 = -2+\sqrt{3}$, $a_2 = -2-\sqrt{3}$

The contour is defined as a unit circle centered at the origin
Therefore, a_2 is not enclosed by the contour
and we take the limit to a_1



$$\oint_C f(z)\,dz = 2\pi i R$$

for $f(z) = \dfrac{z}{(z-a_1)^2(z-a_2)^2} \rightarrow \lim\limits_{z\to a_1} \frac{d}{dz}(z-a_1)^2 f(z) = \lim\limits_{z\to a_1} \frac{d}{dz}\dfrac{z}{(z-a_2)^2}$

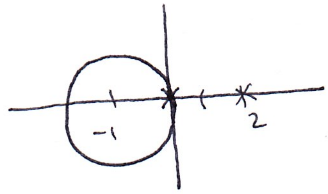
$$= \lim_{z\to a_1} (z-a_2)^{-2} + z(-2(z-a_2)^{-3}) = \lim_{z\to a_1} \frac{-(z+a_2)}{(z-a_2)^3} = \frac{-(-2+\sqrt{3} + -2-\sqrt{3})}{(-2+\sqrt{3}+2+\sqrt{3})^3}$$

$$R = \frac{4}{(2\sqrt{3})^3} = \frac{1}{2\cdot 3^{3/2}}$$

thus $\displaystyle\oint_c f(z)\,dz = 2\pi i \cdot \frac{1}{2\cdot 3^{3/2}}$

$$I = \left(\frac{4}{\cancel{i}}\right)\cancel{2}\pi\cancel{i}\cdot\frac{1}{\cancel{2}\cdot 3^{3/2}} = \frac{4\pi}{3^{3/2}} \rightarrow \boxed{I = \frac{4\pi}{3^{3/2}}}$$

2 a. $f(z) = \dfrac{1}{z(2-z)}$



$|z+1| < 1$

geometric series

$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x|<1$

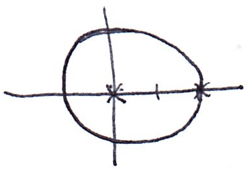
where here, $|x| = |z+1|$

$$f(z) = \frac{1}{z(2-z)} = \frac{1}{2}\left(\frac{1}{z} + \frac{1}{2-z}\right) = \frac{1}{2}\left(\frac{1}{(z+1)-1} + \frac{1}{3-(z+1)}\right)$$

$$= \frac{1}{2}\left(\frac{-1}{1-(z+1)} + \frac{1}{3}\cdot\frac{1}{1-\left(\frac{z+1}{3}\right)}\right) = \frac{1}{2}\left((-1)\sum_{n=0}^{\infty}(z+1)^n + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n\right)$$

$$\boxed{= \frac{1}{2}\sum_{n=0}^{\infty}\left(-1 + \frac{1}{3^{n+1}}\right)(z+1)^n \quad \text{this series is convergent for } |z+1|<1}$$

b. Laurent series about $z=0$



for $0<|z|<2$ $\quad \dfrac{1}{z}\cdot\dfrac{1}{2-z} = \dfrac{1}{2}\cdot\dfrac{1}{z}\cdot\dfrac{1}{1-\frac{z}{2}} = \dfrac{1}{2}\cdot\dfrac{1}{z}\cdot\sum_{n=0}^{\infty}\left(z/2\right)^n$

$$\boxed{= \sum_{n=0}^{\infty} z^{n-1}\frac{1}{2^{n+1}} \quad \text{for } 0<|z|<2}$$

for $|z|>2$ I need $\dfrac{1}{|z|}<2$

$$\frac{1}{z}\cdot\frac{1}{2-z} = \frac{1}{2}\frac{1}{z}\cdot\frac{1}{1-\frac{z}{2}}\cdot\frac{1/z}{1/z} = \frac{1}{2}\cdot\frac{1}{z^2}\cdot\frac{1}{\frac{1}{z}-\frac{1}{2}} = \frac{1}{z^2}\cdot\frac{\cancel{2}\,1}{\frac{2}{z}-1}$$

$$= -\frac{1}{z^2}\cdot\frac{1}{1-2/z} = -\frac{1}{z^2}\cdot\sum\left(\frac{2}{z}\right)^n = \boxed{-\sum_{n=0}^{\infty} 2^n\frac{1}{z^{n+2}} \quad \text{for } |z|>2}$$