18.075 | HW #2 | Colin Kelsall 2/16/17

① i^{2i}

a)

$$i = 0 + i\cdot 1 = \cos\left(\tfrac{\pi}{2}\right) + i\sin\left(\tfrac{\pi}{2}\right) = e^{i\frac{\pi}{2}}$$

$$i^{2i} = \left(e^{i\frac{\pi}{2}}\right)^{2i} = \boxed{e^{-\pi}}$$

b)

$$(z+1)^3 = (z^2-1)^3$$

$$e^{2\pi n i}(z+1)^3 = (z^2-1)^3$$ $z = -1$ is a solution

$$e^{\frac{2}{3}\pi n i}(\cancel{z+1}) = (\cancel{z+1})(z-1)$$

$$e^{\frac{2}{3}\pi n i} = z-1$$

5 roots

$$z = e^{\frac{2}{3}\pi n i} + 1 \qquad n = 0,1,2,3,4,5$$

$$z = \underset{n=0}{2},\ \underset{n=1}{e^{\frac{2}{3}\pi i}+1},\ \underset{n=2}{e^{\frac{4}{3}\pi i}+1},\ \underset{n=3}{e^{2\pi i}+1} = 2,\ \underset{n=4}{e^{\frac{8}{3}\pi i}+1},\ \underset{n=5}{-1}$$

$$z = \underset{\text{repeated}}{2},\ e^{\frac{2}{3}\pi i}+1,\ e^{\frac{4}{3}\pi i}+1,\ e^{\frac{8}{3}\pi i}+1,\ -1$$

$$z = 2,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i+1,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i+1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}+1,\ -1$$

$$\boxed{z = 2,\ -1,\ \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i}$$

2) a)
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$ Laplace Eqn.

$$V(x,y) = xy^2 \qquad \frac{\partial v}{\partial y} = 2xy \quad \frac{\partial^2 v}{\partial y^2} = 2x$$

$$\boxed{0 + 2x = 0}$$ No, this is not satisfied for all x

b) $u = xy^3 + ax^3y$

$$\frac{\partial u}{\partial x} = y^3 + 3ayx^2$$

$$\frac{\partial^2 u}{\partial x^2} = 0 + 6ayx$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial y} = 3xy^2 + ax^3$$

$$6ayx + 6xy = 0$$

$$\frac{\partial^2 u}{\partial y^2} = 6xy + 0$$

$$\boxed{a = -1}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \longrightarrow y^3 - 3yx^2 = \frac{\partial v}{\partial y}$$

$$V = \frac{1}{4}y^4 - \frac{3}{2}y^2x^2 + f_1(x)$$

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \longrightarrow 3xy^2 - x^3 = \frac{-\partial v}{\partial x}$$

$$V = \frac{1}{4}x^4 - \frac{3}{2}x^2y^2 + f_2(y)$$

$$\boxed{V = \frac{1}{4}(x^4 + y^4) - \frac{3}{2}x^2y^2 + c}$$

b (cont.)

$$u = xy^3 - x^3y$$

$$v = \left(\frac{1}{4}(x^4+y^4) - \frac{3}{2}x^2y^2\right) + ci$$

$$f(x,y) = \frac{1}{4}ix^4 + xy^3 - \frac{3}{2}ix^2y^2 - x^3y + \frac{1}{4}iy^4 + ci$$

$$= \frac{1}{4}\left(ix^4 + 4xy^3 - 6ix^2y^2 - 4x^3y + iy^4\right) + ci$$

$$= \frac{i}{4}\left(x^4 + 4ix^3y - 6x^2y^2 - 4xy^3 + y^4\right) + ci$$

$$f(x,y) = \frac{i}{4}(x+iy)^4 + ci$$

$$\boxed{f(z) = \frac{i}{4}z^4 + ci}$$

$$\cos z = \frac{e^{iz} + \frac{1}{e^{iz}}}{2}$$

3) $u = \cos^2 5\theta$ @ $r=3$

$\cos 2x = 2\cos^2 x - 1$

a) $\nabla^2 u = 0$

$\frac{\cos(2x)+1}{2} = \cos^2 x$

$u(3,\theta) = \cos^2 5\theta$

$\left(z+\frac{1}{z}\right)^2 = \left(2\cos(5\theta)\right)^2$ when $z = e^{i5\theta}$

$z^2 = e^{i10\theta}$

$z^2 + 2 + \frac{1}{z^2} = 2^2\cos^2(5\theta)$

$z^2 + \frac{1}{z^2} + 2 = 2^2\cos^2(5\theta)$

$2\cos(10\theta) + 2 = 2^2\cos^2(5\theta)$

$\frac{1}{2}(\cos(10\theta)+1) = \cos^2(5\theta)$

3a) (cont.)

$$u(3,\theta) = \frac{1}{2}(\cos 10\theta + 1) \qquad \nabla^2 u = 0$$

$$u(r,\theta) = \frac{r^{10}}{3^{10}}\left(\frac{\cos 10\theta}{2}\right) + \frac{1}{2}$$

$$\boxed{u(r,\theta) = \frac{r^{10}}{2(3^{10})}\cos 10\theta + \frac{1}{2}}$$

b) $u(2,\theta) = \cos 3\theta$

$u(1,\theta) = 1$

$$u(r,\theta) = \left(ar^3 + \frac{b}{r^3}\right)\cos(3\theta) + (c + d\ln r)$$

$$u(2,\theta) = \left(8a + \frac{b}{8}\right)\cos(3\theta) + c + d\ln 2 = \cos 3\theta$$

$$8a + \frac{b}{8} = 1 \qquad c + d\ln 2 = 0$$

$$u(1,\theta) = (a + b)\cos(3\theta) + c + d\ln(1) = 1$$

$$a + b = 0 \Rightarrow a = -b, \quad 8a + \frac{-a}{8} = 1$$

$$a = \frac{1}{8 - \frac{1}{8}}$$

$$a = \frac{8}{7}$$

$$b = \frac{-8}{7}$$

$$a = \frac{8}{63}$$

$$b = -\frac{8}{63}$$

$$\begin{cases} c + d\ln 1 = 1 \\ c + d\ln 2 = 0 \end{cases}$$

$$1 - d\ln 1 = -d\ln 2$$

$$d = \frac{1}{\ln 1 - \ln 2} = \frac{1}{\ln(\frac{1}{2})}$$

$$c = -\frac{\ln(2)}{\ln(\frac{1}{2})}$$

$$\boxed{\cancel{u(r,\theta) = \frac{8}{7}\left(r^3 - \frac{1}{r^3}\right)\cos(3\theta) - \frac{1}{\ln(\frac{1}{2})}(\ln(2) - 1)}}$$ log r

$$\boxed{u = \frac{8}{63}\left(r^3 - \frac{1}{r^3}\right)\cos 3\theta + 1 - \frac{\log r}{\log 2}}$$