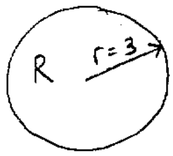
Brad Bachu 18.075 Pset 3 40

1) Let $u$ satisfy the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ in a region $R$.

a) Let $R$ be the disk $x^2 + y^2 < 9$, and $u$ satisfies the boundary condition $u = \cos^2 4\theta$ at $r=3$. WTF $u$.

$$\nabla^2 u = 0$$




$$u(3,\theta) = \cos^2 4\theta$$

We know $\left(z + \frac{1}{z}\right)^n = (2\cos 4\theta)^n$ where $z = e^{i4\theta}$

Consider $n=2$; 
$$2^2 \cos^2 4\theta = \left(z + \frac{1}{z}\right)^2$$
$$= z^2 + 2 + \frac{1}{z^2}$$
$$= z^2 + \frac{1}{z^2} + 2$$
$$= 2\cos 8\theta + 2$$
$$\cos^2 4\theta = \frac{1}{2}\cos 8\theta + \frac{1}{2}$$

Hence, we can rewrite the boundary conditions as:

$$u(3,\theta) = \frac{1}{2}\cos 8\theta + \frac{1}{2}$$

$$\therefore u(r,\theta) = \frac{r^8}{3^8}\frac{\cos 8\theta}{2} + \frac{1}{2}$$

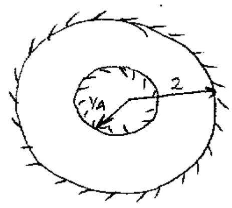
$$\boxed{u(r,\theta) = \frac{r^8 \cos 8\theta}{2 \cdot 3^8} + \frac{1}{2}}$$

b) Let $R$ be the annulus between the circle $x^2+y^2=4$ and the circle $x^2+y^2=\frac{1}{4}$.

Now $u$ satisfies the boundary condition

$u(2,\theta) = \cos 3\theta$

$u(\frac{1}{2},\theta) = 1$




$$u(r,\theta) = \left(ar^3 + \frac{b}{r^3}\right)\cos 3\theta + (c + d\ln r)$$

$$u(2,\theta) = \left(a8 + \frac{b}{8}\right)\cos 3\theta + (c + d\ln 2) = \cos 3\theta \quad (1)$$

$$u(\tfrac{1}{2},\theta) = \left(\frac{a}{8} + 8b\right)\cos 3\theta + (c \cancel{+} - d\ln 2) = 1 \quad (2)$$

From (1); $8a + \frac{b}{8} = 1$

From (2); $\frac{a}{8} + 8b = 0$

$\Rightarrow 8(-8^2 b) + \frac{b}{8} = 1$

$-8^3 b + \frac{b}{8} = 1$

$b\left(\frac{1}{8} - 8^3\right) = 1$

$b = \dfrac{1}{\left(\frac{1}{8} - 8^3\right)}$

$a = -8^2\left(\dfrac{1}{\frac{1}{8} - 8^3}\right)$

$c + d\ln 2 = 0$

$c - d\ln 2 = 1$

$\Rightarrow \cancel{c} - d\ln 2 - d\ln 2 = 1$

$-2d\ln 2 = 1$

$d = \dfrac{-1}{2\ln 2}$

$c = -d\ln 2$

$= -\left(\dfrac{-1}{2\ln 2}\right)\cancel{2}\ln 2$

$c = \dfrac{+1}{2}$

$$\therefore \boxed{u(r,\theta) = \left(\frac{-8^2}{\frac{1}{8} - 8^3} r^3 + \frac{1}{r^3\left(\frac{1}{8} - 8^3\right)}\right)\cos 3\theta \cancel{+} + \frac{1}{2} - \frac{1}{2\ln 2}\ln r}$$

4a) $I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x-2i)(x-3i)(x-4i)}$

$$= \int_{-\infty}^{\infty} \frac{dx}{(x-i)(x+i)(x-2i)(x-3i)(x-4i)}$$

It is simpler to close this contour downstairs. The contribution from the semicircle in the lower half-plane vanishes. If $z$ is a point on the semicircle, $z = e^{i\theta}R$, $-\pi \le \theta \le 0$. When $R$ is very large the contribution of the semicircle is $R/R^5$. Thus this vanishes as $R \to \infty$. With the semicircle included the contour encloses 1 singularity of the integrand located at $-i$.

$$\frac{1}{(x-i)(x+i)(x-2i)(x-3i)(x-4i)} = \frac{1}{120(x+i)} + \frac{1}{12(x-i)} - \frac{1}{6(x-2i)} - \frac{1}{8(x-3i)} + \frac{1}{30(x-4i)}$$

$$Res(-i) = \frac{1}{-120i}$$

$$\therefore I = -2\pi i \, Res(-i)$$

$$= -2\pi i\left(\frac{1}{-120i}\right) = \frac{\pi}{60}$$

4d) $I = \int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2}\, d\theta \quad (a>b>0)$

Change of variable : $z \equiv e^{i\theta}$

As $\theta$ varies from 0 to $2\pi$, $z$ traverses in the counter clockwise direction the unit circle with the center at the origin.

$$dz = e^{i\theta} i\, d\theta$$

$$d\theta = \frac{dz}{iz}$$

We also have $\cos\theta = \frac{1}{2}(z+z^{-1})$

$$\frac{1}{(a+b\cos\theta)^2}\cdot d\theta = \frac{dz}{iz\left(a+b\frac{1}{2}(z+z^{-1})\right)^2}$$

$$= \frac{dz}{iz\left(\frac{1}{2}(2a+b(z+z^{-1})\right)^2}$$

$$= \frac{dz}{iz\frac{1}{4}\left(\frac{1}{z}(bz^2+2a+b)\right)^2}$$

$$= \frac{z\,4dz}{i(bz^2+2a+b)^2} = \frac{z\,4dz}{ib^2(z^2+2\%+1)^2}$$

$$= \frac{z\,4dz}{b^2 i(z-z_0)^2(z-z_1)^2}$$

Since only $z_0$ is in $C$, we have

$$\frac{4}{b^2 i}\oint_C \frac{z}{(z-z_0)^2(z-z_1)^2}\cdot dz = 2\pi i \operatorname{Res}(z_0)$$

$z_0$ is a pole of order 2

$$\operatorname{Res}(z_0) = \lim_{z\to z_0}\frac{d}{dz}(z-z_0)^2 f(z) = \lim_{z\to z_0}\frac{d}{dz}\frac{z}{(z-z_1)^2}$$

$$= \lim_{z\to z_0} -\frac{z+z_1}{(z-z_1)^3} = \frac{ab^2}{4(a^2-b^2)^{3/2}}$$

$$\therefore \frac{4}{b^2 i}\oint\frac{z\,dz}{(z-z_0)^2(z-z_1)^2} = \frac{4}{b^2 i}2\pi i\cdot\frac{ab^2}{4(a^2-b^2)^{3/2}}$$

$$= \frac{2\pi a}{(a^2-b^2)^{3/2}} \quad \checkmark$$

$2a + bz + \frac{b}{z}$

$\frac{1}{z}(2az + bz^2 + b)$

$bz^2 + 2a + 1 = 0$

$$z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b}$$

$$= -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - \frac{1}{b}1}$$

$= z_0, z_1$

$$z_0 = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2}-1}$$

$$z_1 = -\frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1}$$

$z_0 + z_1 = -2a/b$

$$z_0 - z_1 = 2\sqrt{\frac{a^2}{b^2}-1}$$

$$\frac{z_0+z_1}{(z_0-z_1)^3} = \frac{-2a/b}{2^3\left(\sqrt{\frac{a^2}{b^2}-1}\right)^3}$$

$$= \frac{-a}{2^2 b\left(\frac{1}{b}(a^2-b^2)^{1/2}\right)^3}$$

$$= \frac{-a}{4b^{-2}(a^2-b^2)^{3/2}}$$

$$= \frac{-ab^2}{4(a^2-b^2)^{3/2}}$$