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18.075 Problem set 2

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Problem 1. Part a. We want to find the value of i^{2i}. For this we use the fact that

$$i = 0 + i1 = \cos(\pi/2) + i\sin(\pi/2) = e^{i\pi/2}$$

So we have that

$$i^{2i} = \left(e^{i\pi/2}\right)^{2i} = e^{i^2\pi} = e^{-\pi}$$

√

Therefore, the value of i^{2i} is $e^{-\pi}$.

Part b. We want to find the roots of

$$(z+1)^4 = (z^2-1)^4$$

We start by multiplying the left side by 1 in the polar form $e^{2\pi ni}$ and solving the equation

$$\begin{aligned} e^{2\pi ni}(z+1)^4 &= (z^2-1)^4 \\ e^{2\pi ni/4}(z+1) &= (z^2-1) \\ e^{in\pi/2}(z+1) &= (z-1)(z+1) \\ e^{in\pi/2} &= (z-1) \\ z &= e^{in\pi/2}+1 \end{aligned}$$

Note that we assumed $z \neq -1$. This gives us four distinct roots

$$z = e^{in\pi/2}+1 \quad \text{for } n = 0, 1, 2, 3$$

The value $z = -1$ also solves the equation, so we now have 5 roots. Although this is a polynomial of degree 8, some of its roots are repeated, so there are only 5 distinct roots. All the roots in cartesian form are the following

$$z = 2,\ 1+i,\ 0,\ 1-i,\ -1$$

√

Problem 2. First we want to find the real and imaginary parts of e^{iz}, where $z = x + iy$. Expanding this we have

$$e^{iz} = e^{ix+i^2y} = e^{ix}e^{-y} = (\cos(x) + i\sin(x))e^{-y} = e^{-y}\cos(x) + ie^{-y}\sin(x)$$

Therefore, the real and imaginary parts are

$$\mathrm{Re}(e^{iz}) = e^{-y}\cos(x) \qquad \mathrm{Im}(e^{iz}) = e^{-y}\sin(x)$$

Now we want to find the real and imaginary parts of $\cos(z)$. For this we use the result from the previous calculation and the fact that $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$. Computing this expression we get

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}(e^{-y}\cos(x) + ie^{-y}\sin(x) + e^{y}\cos(-x) + ie^{y}\sin(-x))$$

So the real and imaginary parts are

$$\mathrm{Re}(\cos(z)) = \frac{1}{2}\cos(x)(e^{-y} + e^{y}) \qquad \mathrm{Im}(\cos(z)) = \frac{1}{2}\sin(x)(e^{-y} - e^{y})$$

Finally, we want to find the values for which the real part of e^{iz} is equal to $\cos(z)$.

$$e^{-y}\cos(x) = \frac{1}{2}\cos(x)(e^{-y} + e^{y}) + i\frac{1}{2}\sin(x)(e^{-y} - e^{y})$$

Note that the left side of the equation is real, so the imaginary part on the right must be zero. Also, the real part must be equal so $\frac{1}{2}(e^{-y} + e^{y}) = e^{-y}$. Both of these conditions are satisfied only when $y = 0$. Therefore, the real part of e^{iz} is equal to $\cos(z)$ only when z is real.

not complete

Because the statement of the problem was slightly ambiguous I will also check for what values of z the real part of e^{iz} is equal to the real part of $\cos(z)$. In this case we have

$$\begin{aligned} e^{-y}\cos(x) &= \frac{1}{2}\cos(x)(e^{-y} + e^{y}) \\ e^{-y} &= \frac{1}{2}(e^{-y} + e^{y}) \\ e^{-y} &= e^{y} \\ -y &= y \end{aligned}$$

So this is satisfied when $y = 0$. Note that we ignored the case when $\cos(x) = 0$ which also solves the equation. Therefore, the real part of e^{iz} is equal to

$\cos x$ and $\sin x$ cannot both be 0. Hence either $e^{-y} = \frac{1}{2}(e^{-y} + e^{y})$ or $e^{-y} = e^{y}$. In both cases it implies $y = 0$

the real part of $\cos(z)$ only when z is real or when $z = (2n-1)\pi/2 + iy$ for integer n and arbitrary y.

Problem 3. We want to find the values of a for which the function $u = xy^2 + ax^3$ is the real part of an analytic function. The partial derivatives of this function are

$$\frac{\partial u}{\partial x} = y^2 + 3ax^2 \qquad \frac{\partial^2 u}{\partial x^2} = 6ax$$
$$\frac{\partial u}{\partial y} = 2xy \qquad \frac{\partial^2 u}{\partial y^2} = 2x$$

From class we know that the real part of an analytic function must satisfy the Laplace equation. So we find the value of a such that u satisfies such equation.

$$\begin{aligned} \nabla^2 u &= 0 \\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\ 6ax + 2x &= 0 \\ x &= -\frac{1}{3} \end{aligned}$$

Therefore, for u to be the real part of an analytic function then $a = -\frac{1}{3}$.

Now we want to find the imaginary part v of this analytic function. We do this by using the Cauchy-Riemann equations. We start by doing the following

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} &= y^2 - x^2 \\ v &= \int (y^2 - x^2)dy + g(x) \\ &= \frac{y^3}{3} - x^2 y + g(x) \end{aligned}$$

where g is some function of x. To find this function we use the other equation

$$\begin{aligned}
-\frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y} \\
2xy - \frac{\partial g}{\partial x} &= 2xy \\
\frac{\partial g}{\partial x} &= 0 \\
g &= \int 0 dx + c \\
&= c
\end{aligned}$$

where c is an arbitrary constant. Therefore, the imaginary part of this analytic function is

$$v = \frac{y^3}{3} - x^2 y + c$$

So the complete analytic function is

$$\begin{aligned}
f(x, y) &= xy^2 - \frac{1}{3}x^3 + i\frac{y^3}{3} - ix^2 y + ic = -\frac{1}{3}(x + iz)^3 + ic \\
f(z) &= -\frac{1}{3}z^3 + ic
\end{aligned}$$