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18.075 Problem Set 1

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1. (1a). $x(1+x)y' + \frac{1+x}{2}y = \sqrt{x} \Rightarrow x(1+x)(D+\frac{1}{2x})y = \sqrt{x} \Rightarrow (D+\frac{1}{2x})y = \frac{1}{\sqrt{x}(x+1)}$

This is of the form $(D+p(x))y = q(x)$. This has the solution $y(x) = e^{-P(x)}\left(\int_0^x e^{P(x')}q(x')dx' + c\right)$.

$P(x) = \int_0^x p(x)dx' = \int_0^x \frac{1}{2x'}dx = \frac{1}{2}\ln(x)$. Then $e^{\frac{1}{2}\ln(x)} = \sqrt{x}$ and $y(x) = \frac{1}{\sqrt{x}}\left(\int_0^x \sqrt{x'} \cdot \frac{1}{\sqrt{x'}(x'+1)}dx' + c\right)$

$\boxed{y(x) = \frac{1}{\sqrt{x}}(\ln(x+1) + c)}$ To check: $x(1+x)\left(\frac{1}{\sqrt{x}}\frac{1}{x+1} - \frac{1}{2x}\frac{1}{\sqrt{x}}\ln(1+x)\right) + \frac{1+x}{2}\frac{1}{\sqrt{x}}\ln(1+x) = \sqrt{x}$

(1c) $y' = \frac{1}{x}y + xy^2$. Let $u \equiv \frac{1}{y}$, $u' = -u^2y'$. Then the differential equation becomes

$-\frac{u'}{u^2} = \frac{1}{x}\cdot\frac{1}{u} + \frac{x}{u^2}$, which can be rewritten $(D+\frac{1}{x})u = -x$. Following the same procedure as above,

$u(x) = \frac{1}{x}\left(\int_0^x x' \cdot -x'dx' + c'\right) = \frac{1}{x}\left(\int_0^x -x'^2dx' + c'\right) = \frac{1}{x}\left(-\frac{x^3}{3} + c'\right) = \frac{-x^3+c}{3x}$ $(3c' = -c)$

since $y = \frac{1}{u}$, $\boxed{y(x) = \frac{-3x}{x^3+c}}$. To check: $\frac{-3(x^3+c)+3x(3x^2)}{(x^3+c)^2} = \frac{-3}{x^3+c} + \frac{9x^3}{(x^3+c)^2}$. Also $y \equiv 0$ is a solution

2f) $y''' + y'' - y' - y = xe^x + 7\cosh x$. First, find the complementary solution: Let $Y = e^{mx}$. Then

$m^3+m^2-m-1 = 0 \Rightarrow (m^2-1)(m+1) = 0 \Rightarrow (m+1)^2(m-1) = 0$. Two of the complementary solutions

are e^x and e^{-x}. For the third (to account for the repeated root $m=-1$), now let $Y = e^{-x}v$.

Then $(D+1)^2(e^{-x}v) = 0$ or $e^{-x}D^2v = 0$ which gives $v = (c_1 + c_2x)$. So the full complementary

solution is $\underline{Y = (c_1+c_2x)e^{-x} + c_3e^x}$. Now, find a particular solution:

$y_P = \frac{1}{D^3+D^2-D-1}(xe^x + 7\cosh x)$. First, $y_{P_1} = \frac{1}{D^3+D^2-D-1}e^x x = \frac{1}{(D+1)^2(D-1)}e^x x$

$= \left(\frac{1}{4(D-1)} - \frac{1}{4(D+1)} - \frac{1}{2(D+1)^2}\right)e^x x = e^x\left(\frac{1}{4}\frac{1}{D} - \frac{1}{4}\frac{1}{(D+2)} - \frac{1}{2}\frac{1}{(D+2)^2}\right)x$

$\frac{1}{D+2}x = \frac{1}{2}\frac{1}{1+D/2}x = \frac{1}{2}\left(1 - \frac{D}{2} + \frac{D^2}{4} + \cdots\right)x = \frac{1}{2}(x - \frac{1}{2}) = \frac{1}{2}x$ + term in Y (c_3e^x)

$\frac{1}{D+2}\frac{1}{2}x = \frac{1}{4}x$.

$y_{P_1} = e^x\left(\frac{1}{8}x^2 - \frac{1}{4}\cdot\frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{4}x\right) = \underline{e^x\left(\frac{x^2}{8} - \frac{x}{4}\right)}$

Now, $y_{P_2} = \frac{7}{2}\left(\frac{1}{4(D-1)} - \frac{1}{4(D+1)} - \frac{1}{2(D+1)^2}\right)(e^x + e^{-x}) = \frac{7}{2}e^x \cdot \frac{1}{4D}\cdot 1 - \frac{7}{2}e^{-x}\left(\frac{1}{4}\frac{1}{D} - \frac{1}{2}\frac{1}{D^2}\right)$

$= \frac{7}{2}\left(\frac{e^x \cdot x}{4} - e^{-x}\left(\frac{x}{8} - \frac{x^2}{4}\right)\right) = \underline{\frac{7}{8}xe^x - \frac{7}{8}x^2e^{-x}}$

So, all in all:

$$y = \frac{1}{8}x^2e^x - \frac{1}{4}xe^x + \frac{7}{8}xe^x - \frac{7}{8}x^2e^{-x} + (c_1+c_2x)e^{-x} + c_3e^x$$

$$\boxed{y = \frac{1}{8}x^2e^x - \frac{7}{8}x^2e^{-x} + \frac{5}{8}xe^x + (c_1+c_2x)e^{-x} + c_3e^x} \quad V$$

2. $y'' - y = x^2 + \cos 2x + e^x$

complementary solution: $y''-y=0$, $(D^2-1)e^{mx}=0$, $m=\pm 1 \Rightarrow Y = ae^x + be^{-x}$

particular solution: $(D^2-1)y_P = x^2 + \cos 2x + e^x$. $\quad y_{P_1} = \frac{-1}{1-D^2}x^2 = -(1+D^2+\cdots)x^2 = -x^2-2$.

$y_{P_2} = \frac{1}{D^2-1}\cos 2x = \frac{1}{-4-1}\cos 2x = \frac{-1}{5}\cos 2x$ $\quad y_{P_3} = \frac{1}{D^2-1}e^x = \frac{1}{(D+1)(D-1)}e^x = \left(\frac{1}{2}\frac{1}{D-1} - \frac{1}{2}\frac{1}{D+1}\right)e^x$ a (in Y)

$= e^x \cdot \frac{1}{2} \cdot \frac{1}{D} \cdot 1 = \frac{1}{2}xe^x$. General solution: $\boxed{y = -x^2-2-\frac{1}{5}\cos 2x + \frac{1}{2}xe^x + ae^x + be^{-x}}$

With initial conditions $y(0)=0$, $y'(0)=1$: $\quad y' = -2x + \frac{2}{5}\sin 2x + \frac{1}{2}e^x(x+1) + ae^x - be^{-x}$

$y(0) = -2 - \frac{1}{5} + a + b$. $\quad y'(0) = \frac{1}{2} + a - b$. So the solution is

$a+b = \frac{11}{5}$, $a-b = \frac{1}{2} \Rightarrow a = \frac{27}{20}$, $b = \frac{17}{20}$ $\quad \boxed{y = -x^2-2-\frac{1}{5}\cos 2x + \frac{1}{2}xe^x + \frac{27}{20}e^x + \frac{17}{20}e^{-x}}$ V

3. $y' = x - y^2$. Let $y(x) \equiv -\frac{u'(x)}{d(x)u(x)}$ which will give $u''(x) - xu(x) = 0$ or $(D^2-x)u = 0$. This is linear ~~is linear~~ ok

It is also, after some googling, the Airy equation.