31. Phase portraits in two dimensions

This section presents a very condensed summary of the behavior of two-dimensional linear systems, followed by a catalogue of linear phase portraits. A much richer understanding of this gallery can be achieved using the Mathlets Linear Phase Portraits: Cursor Entry and Linear Phase Portraits: Matrix Entry.

31.1. Phase portraits and eigenvectors. It is convenient to represent the solutions of an autonomous system \( \dot{x} = f(x) \) (where \( x = \begin{bmatrix} x \\ y \end{bmatrix} \)) by means of a phase portrait. The \( x, y \) plane is called the phase plane (because a point in it represents the state or phase of a system). The phase portrait is a representative sampling of trajectories of the system. A trajectory is the directed path traced out by a solution. It does not include information about the time at which solutions pass through various points (which will depend upon when the clock was set), nor does it display the speed at which the solution passes through the point—only the direction of travel. Still, it conveys essential information about the qualitative behavior of solutions of the system of equations.

The building blocks for the phase portrait of a general system will be the phase portraits of homogeneous linear constant coefficient systems: \( \dot{x} = Ax \), where \( A \) is a constant square matrix. Notice that this equation is autonomous!

The phase portraits of these linear systems display a startling variety of shapes and behavior. We’ll want names for them, and the names I’ll use differ slightly from the names used in the book and in some other sources.

One thing that can be read off from the phase portrait is the stability properties of the system. A linear autonomous system is unstable if most of its solutions tend to infinity with time. (The meaning of “most” will become clearer below.) It is asymptotically stable if all of its solutions tend to 0 as \( t \) goes to \( \infty \). Finally it is neutrally stable if none of its solutions tend to infinity with time but most of them do not tend to zero either. It is an interesting fact that any linear autonomous system exhibits one of these three behaviors.

The characteristic polynomial of a square matrix \( A \) is defined to be

\[
p_A(s) = \det(A - sI).
\]
If $A$ is $n \times n$, this polynomial has the following form:

$$p_A(s) = (-s)^n + (\text{tr}A)(-s)^{n-1} + \cdots + (\text{det}A),$$

where the dots represent less familiar combinations of the entries of $A$.

When $A$ is $2 \times 2$, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, this reads

$$p_A(s) = s^2 - (\text{tr}A)s + (\text{det}A).$$

We remind the reader that in this case, when

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$\text{tr}A = a + d, \quad \text{det}A = ad - bc.$$

From the eigenvalues we may reconstruct $\text{tr}A$ and $\text{det}A$, since

$$p_A(s) = (s - \lambda_1)(s - \lambda_2) = s^2 - (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2$$

implies

$$\text{tr}A = \lambda_1 + \lambda_2, \quad \text{det}A = \lambda_1\lambda_2.$$

Thus giving the trace and the determinant is equivalent to giving the pair of eigenvalues.

Recall that the general solution to a system $\dot{x} = Ax$ is usually of the form $c_1e^{\lambda_1 t}\alpha_1 + c_2e^{\lambda_2 t}\alpha_2$, where $\lambda_1, \lambda_2$ are the eigenvalues of the matrix $A$ and $\alpha_1, \alpha_2$ are corresponding nonzero eigenvectors. The eigenvalues by themselves usually describe most of the gross structure of the phase portrait.

There are two caveats. First, this is not necessarily the case if the eigenvalues coincide. In two dimensions, when the eigenvalues coincide one of two things happens. (1) The complete case. Then $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$, every vector is an eigenvector (for the eigenvalue $\lambda_1 = \lambda_2$), and the general solution is $e^{\lambda_1 t}\alpha$ where $\alpha$ is any vector. (2) The defective case. (This covers all the other matrices with repeated eigenvalues, so if you discover your eigenvalues are repeated and you are not diagonal, then you are defective.) Then there is (up to multiple) only one eigenvector, $\alpha_1$, and the general solution is $x = e^{\lambda_1 t}(c_1\alpha_1 + c_2(t\alpha_1 + \beta))$, where $\beta$ is a vector such that $(A - \lambda_1 I)\beta = \alpha_1$. (Such a vector $\beta$ always exists in this situation, and is unique up to addition of a multiple of $\alpha_1$.)

The second caveat is that the eigenvalues may be non-real. They will then form a complex conjugate pair. The eigenvectors will also be non-real, and if $\alpha_1$ is an eigenvector for $\lambda_1$ then $\alpha_2 = \overline{\alpha_1}$ is an eigenvector
for $\lambda_2 = \lambda_1$. Independent real solutions may be obtained by taking the
real and imaginary parts of either $e^{\lambda_1 t} \alpha_1$ or $e^{\lambda_2 t} \alpha_2$. (These two have the
same real parts and their imaginary parts differ only in sign.) This will
give solutions of the general form $e^{at}$ times a vector whose coordinates
are linear combinations of $\cos(\omega t)$ and $\sin(\omega t)$, where the eigenvalues
are $a \pm i\omega$.

Each of these caveats represents a failure of the eigenvalues by themselves
to determine major aspects of the phase portrait. In the case of
repeated eigenvalue, you get a defective node or a star node, depending
upon whether you are in the defective case or the complete case. In the
case of non-real eigenvalues you know you have a spiral (or a center, if
the real part is zero); you know whether it is stable or unstable (look
at the sign of the real part of the eigenvalues); but you do not know
from the eigenvalues alone which way the spiral is spiraling, clockwise
or counterclockwise.

31.2. The (tr, det) plane and structural stability. We are now con-
fronted with a large collection of autonomous systems, the linear two-
dimensional systems $\dot{x} = Ax$. This collection is parametrized by the
four entries in the matrix. We have understood that much of the be-
behavior of such a system is determined by two particular combinations
of these four parameters, namely the trace and the determinant.

So we will consider now an entire plane with coordinates $(T, D)$. Whenever we pick a point on this plane, we will be considering the
linear autonomous systems whose matrix has trace $T$ and determinant
$D$.

Such a matrix is not well-defined. For given $(T, D)$ there are always
infinitely many matrices $A$ with $\text{tr}A = T$ and $\text{det}A = D$. One example
is the “companion matrix,”

$$A = \begin{bmatrix} 0 & 1 \\ -D & T \end{bmatrix}.$$ 

This is a particularly important example, because it represents
the system corresponding to the LTI equation $\ddot{x} - T \dot{x} + Dx = 0$, via $y = \dot{x}$. (I’m sorry about the notation here. $T$ and $D$ are just numbers; $Dx$
do not signify the derivative of $x$.)

The $(T, D)$ plane divides into several parts according to the appearance
of the phase portrait of the corresponding matrices. The important regions are as follows.
If $D < 0$, the eigenvalues are real and of opposite sign, and the phase portrait is a saddle (which is always unstable).

If $0 < D < T^2/4$, the eigenvalues are real, distinct, and of the same sign, and the phase portrait is a node, stable if $T < 0$, unstable if $T > 0$.

If $0 < T^2/4 < D$, the eigenvalues are neither real nor purely imaginary, and the phase portrait is a spiral, stable if $T < 0$, unstable if $T > 0$.

These three regions cover the whole of the $(T, D)$ except for the curves separating them from each other, and so are them most commonly encountered and the most important cases. Suppose I have a matrix $A$ with $(\text{tr}A, \det A)$ in one of these regions. If someone kicks my matrix, so that its entries change slightly, I don’t have to worry; if the change was small enough, the new matrix will be in the same region and the character of the phase portrait won’t have changed very much. This is a feature known as “structural stability.”

The remainder of the $(T, D)$ plane is populated by matrices exhibiting various other phase portrait types. They are structurally unstable, in the sense that arbitrarily small perturbations of their entries can, and almost always will, result in a matrix with phase portrait of a different type. For example, when $0 < D$ and $T = 0$, the eigenvalues are purely imaginary, and the phase portrait is a center. But most perturbations of such a matrix will result in one whose eigenvalues have nonzero real part and hence whose phase portrait is a spiral.

31.3. The portrait gallery. Now for the dictionary of phase portraits. In the pictures which accompany these descriptions some elements are necessarily chosen at random. For one thing, most of the time there will be two independent eigenlines (i.e., lines through the origin made up of eigenvectors). Below, if these are real they will be the lines through $\alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\alpha_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If there is only one eigendirection (this only happens if $\lambda_1 = \lambda_2$ and is then called the “defective case”) it will be the line through $\alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If they are not real, they are conjugate to each other and hence distinct. The question of how they influence the phase portrait is more complex and will not be addressed.
Name: Spiral.

**Eigenvalues:** Neither real nor purely imaginary: $0 \neq \frac{\text{tr}^2}{4} < \text{det}$. 

**Stability:** Stable if $\text{tr} < 0$, unstable if $\text{tr} > 0$. 

\[ x' = -2x + 5y \\
\]
\[ y' = -2x + 4y \]

Name: Node.

**Eigenvalues:** Real, same sign: $0 < \text{det} < \frac{\text{tr}^2}{4}$. 

**Stability:** Stable if $\text{tr} < 0$, unstable if $\text{tr} > 0$. 

\[ x' = 2x - y \\
\]
\[ y' = y \]
Name: Saddle.

Eigenvalues: Real, opposite sign: $\det < 0$.

Stability: Unstable.

\[
\begin{align*}
x' &= x - 2y \\
y' &= -y
\end{align*}
\]

Name: Center.

Eigenvalues: Purely imaginary, nonzero: $\text{tr} = 0, \det > 0$.

Stability: Neutrally stable.

\[
\begin{align*}
x' &= 2x - 3y \\
y' &= 2x - 2y
\end{align*}
\]
Name: Defective Node.

Eigenvalues: Repeated (hence real) but nonzero: \( \det = \frac{\text{tr}^2}{4} > 0; \) defective.

Stability: Stable if \( \text{tr} < 0, \) unstable if \( \text{tr} > 0. \)

\[
\begin{array}{ccc}
1 & 2 & 3 \\
2 & 4 & 5 \\
3 & 5 & 6 \\
4 & 6 & 7 \\
5 & 7 & 8 \\
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7 & 9 & 10 \\
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9 & 11 & 12 \\
10 & 12 & 13 \\
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95 & 97 & 98 \\
96 & 98 & 99 \\
97 & 99 & 100
\end{array}
\]

Name: Star Node.

Eigenvalues: Repeated (hence real) but nonzero; complete: \( \det = \frac{\text{tr}^2}{4} > 0. \)

Stability: Stable if \( \text{tr} < 0, \) unstable if \( \text{tr} > 0. \)
Name: Degenerate: Comb.

Eigenvalues: One zero (hence both real).

Stability: Stable if \( \text{tr} < 0 \), unstable if \( \text{tr} > 0 \).

Name: Degenerate: Parallel Lines.

Eigenvalues: Both zero: \( \text{tr} = \text{det} = 0 \); defective.

Stability: Unstable.
Name: Degenerate: Everywhere fixed; $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Eigenvalues: Both zero: $\text{tr} = \text{det} = 0$; complete.

Stability: Neutrally stable.

(No picture; every point is a constant solution.)