28. AMPLITUDE RESPONSE AND THE POLE DIAGRAM

We have seen in Section 10 that the analysis of the system response of an LTI operator to a sinusoidal signal generalizes to the case of a signal of the form $Ae^{at}\cos(\omega t - \phi)$. When $\phi = 0$, one considers an associated complex equation, with input signal given by $Ae^{(a+i\omega)t}$, and applies the Exponential Response Formula.

The Mathlet `Amplitude Response and Pole Diagram` illustrates this situation.

We will consider again the behavior of the spring/mass/dashpot system, in its complex form

$$m\ddot{z} + b\dot{z} + kz = kAe^{st}$$

where s is a complex constant. The Exponential Response Formula gives the exponential solution

(1) $$z_p = AW(s)e^{st}$$

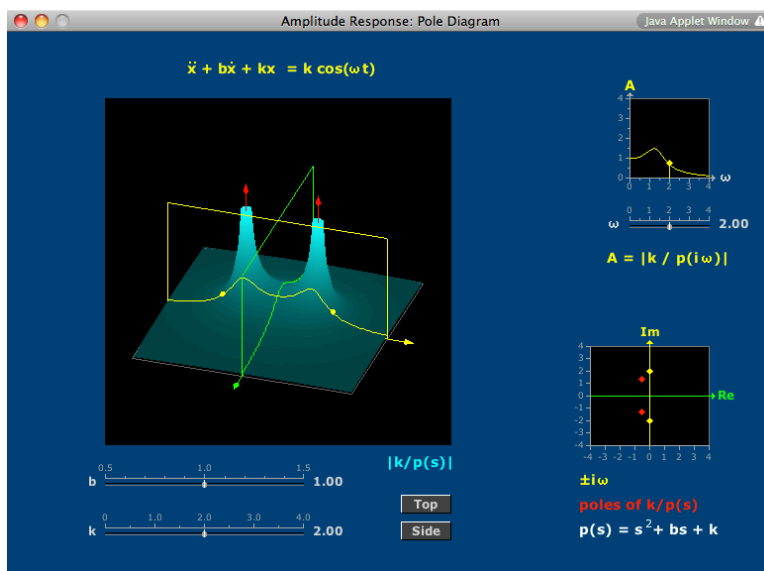where $W(s)$ is the "transfer function"

$$W(s) = \frac{k}{p(s)}$$

FIGURE 16. Screen shot of `Amplitude Response: Pole Diagram`

The input signal gets multiplied by the complex number $W(s)$. This number has a magnitude and an argument. We will focus entirely on

the magnitude. When it is large, the system response is that much larger than the input signal. When it is small, the system response is that much smaller. The real number $|W(s)|$ is called the **gain**.

Now imagine the graph of $|W(s)|$. It is a tent-like surface, suspended over the complex plane. There are places where $|W(s)|$ becomes infinite—at the roots of the characteristic polynomial $ms^2 + bs + k$. These are called **poles** of $W(s)$. The graph flares up to infinite height above the poles of $W(s)$, which may be why they are called poles! The altitude above $s = a + i\omega$ has an interpretation as the "gain" of the system when responding to an input signal of the form $e^{at} \cos(\omega t)$. When s coincides with a pole, the system is in resonance; there is no solution of the form $gAe^{at} \cos(\omega t - \phi)$, but rather one of the form t times that expression.

Near to the poles, the gain is large, and it falls off as s moves away from the poles.

The case of sinusoidal input occurs when s is on the imaginary axis. Imagine wall rising from the imaginary axis. It intersects the graph of $|W(s)|$ in a curve. That curve represents $|W(i\omega)|$ as ω varies over real numbers. This is precisely the "Bode plot," the amplitude frequency response curve.

The `Amplitude Response and Pole Diagram` Mathlet shows this well. (The program chooses $m = 1$.) The left hand window is a 3-D display of the graph of $|W(s)|$. Moving the cursor over that window will reorient the picture. At lower right is the pole diagram of $W(s)$, and above it is the amplitude response curve.

You can see the geometric origin of near resonance: what is happening is that the part of the graph of $|W(s)|$ lying over the imaginary axis moves up over the shoulder of the "volcano" surrounding one of the poles of $W(s)$.