14. The exponential shift law

This section explains a method by which an LTI equation with input signal of the form $e^{rt}q(t)$ can be replaced by a simpler equation in which the input signal is just $q(t)$.

14.1. **Exponential shift.**

The calculation (10.1)

$$p(D)e^{rt} = p(r)e^{rt} \tag{1}$$

extends to a formula for the effect of the operator $p(D)$ on a product of the form $e^{rt}u$, where u is a general function. This is useful in solving $p(D)x = f(t)$ when the input signal is of the form $f(t) = e^{rt}q(t)$.

The formula arises from the product rule for differentiation, which can be written in terms of operators as

$$D(vu) = v\,Du + (Dv)u.$$

If we take $v = e^{rt}$ this becomes

$$D(e^{rt}u) = e^{rt}Du + re^{rt}u = e^{rt}(Du + ru)\,.$$

Using the notation I for the identity operator, we can write this as

$$D(e^{rt}u) = e^{rt}(D + rI)u. \tag{2}$$

If we apply D to this equation again,

$$D^2(e^{rt}u) = D(e^{rt}(D + rI)u) = e^{rt}(D + rI)^2u\,,$$

where in the second step we have applied (2) with u replaced by $(D + rI)u$. This generalizes to

$$D^k(e^{rt}u) = e^{rt}(D + rI)^k u.$$

The final step is to take a linear combination of D^k's, to form a general LTI operator $p(D)$. The result is the

Exponential Shift Law:

$$\boxed{p(D)(e^{rt}u) = e^{rt}p(D + rI)u} \tag{3}$$

The effect is that we have pulled the exponential outside the differential operator, at the expense of changing the operator in a specified way.

14.2. **Product signals.** We can exploit this effect to solve equations of the form

$$p(D)x = e^{rt}q(t)\,,$$

by a version of the method of variation of parameter: write $x = e^{rt}u$, apply $p(D)$, use (3) to pull the exponential out to the left of the operator, and then cancel the exponential from both sides. The result is

$$p(D + rI)u = q(t)\,,$$

a new LTI ODE for the function u, one from which the exponential factor has been eliminated.

Example 14.2.1. Find a particular solution to $\ddot{x} + \dot{x} + x = t^2e^{3t}$.

With $p(s) = s^2 + s + 1$ and $x = e^{3t}u$, we have

$$\ddot{x} + \dot{x} + x = p(D)x = p(D)(e^{3t}u) = e^{3t}p(D + 3I)u\,.$$

Set this equal to t^2e^{3t} and cancel the exponential, to find

$$p(D + 3I)u = t^2$$

This is a good target for the method of undetermined coefficients (Section 11). The first step is to compute

$$p(s + 3) = (s + 3)^2 + (s + 3) + 1 = s^2 + 7s + 13\,,$$

so we have $\ddot{u} + 7\dot{u} + 13u = t^2$. There is a solution of the form $u_p = at^2 + bt + c$, and we find it is

$$u_p = (1/13)t^2 - (14/13^2)t + (85/13^3)\,.$$

Thus a particular solution for the original problem is

$$x_p = e^{3t}((1/13)t^2 - (14/13^2)t + (85/13^3))\,.$$

Example 14.2.2. Find a particular solution to $\dot{x} + x = te^{-t}\sin t$.

The signal is the imaginary part of $te^{(-1+i)t}$, so, following the method of Section 10, we consider the ODE

$$\dot{z} + z = te^{(-1+i)t}\,.$$

If we can find a solution z_p for this, then $x_p = \operatorname{Im} z_p$ will be a solution to the original problem.

We will look for z of the form $e^{(-1+i)t}u$. The Exponential Shift Law (3) with $p(s) = s + 1$ gives

$$\begin{aligned}\dot{z} + z = (D + I)(e^{(-1+i)t}u) &= e^{(-1+i)t}((D + (-1 + i)I) + I)u \\ &= e^{(-1+i)t}(D + iI)u.\end{aligned}$$

When we set this equal to the right hand side we can cancel the exponential:

$$(D + iI)u = t$$

or $\dot{u} + iu = t$. While this is now an ODE with *complex* coefficients, it's easy to solve by the method of undetermined coefficients: there is a solution of the form $u_p = at + b$. Computing the coefficients, $u_p = -it + 1$; so $z_p = e^{(-1+i)t}(-it + 1)$.

Finally, extract the imaginary part to obtain x_p:

$$z_p = e^{-t}(\cos t + i \sin t)(-it + 1)$$

has imaginary part

$$x_p = e^{-t}(-t \cos t + \sin t).$$

14.3. **Summary.** The work of this section and the previous two can be summarized as follows: Among the responses by an LTI system to a signal which is polynomial times exponential (or a linear combination of such) there is always one which is again a linear combination of functions which are polynomial times exponential. By the magic of the complex exponential, sinusoidal factors are included in this.