14. THE EXPONENTIAL SHIFT LAW

This section explains a method by which an LTI equation with input signal of the form $e^{rt}q(t)$ can be replaced by a simpler equation in which the input signal is just $q(t)$.

14.1. Exponential shift. The calculation (10.1)

(1) \[ p(D)e^{rt} = p(r)e^{rt} \]

extends to a formula for the effect of the operator $p(D)$ on a product of the form $e^{rt}u$, where $u$ is a general function. This is useful in solving $p(D)x = f(t)$ when the input signal is of the form $f(t) = e^{rt}q(t)$.

The formula arises from the product rule for differentiation, which can be written in terms of operators as

\[ D(vu) = vDu + (Dv)u. \]

If we take $v = e^{rt}$ this becomes

\[ D(e^{rt}u) = e^{rt}Du + re^{rt}u = e^{rt}(Du + ru). \]

Using the notation $I$ for the identity operator, we can write this as

(2) \[ D(e^{rt}u) = e^{rt}(D + rI)u. \]

If we apply $D$ to this equation again,

\[ D^2(e^{rt}u) = D(e^{rt}(D + rI)u) = e^{rt}(D + rI)^2u, \]

where in the second step we have applied (2) with $u$ replaced by $(D + rI)u$. This generalizes to

\[ D^k(e^{rt}u) = e^{rt}(D + rI)^k u. \]

The final step is to take a linear combination of $D^k$'s, to form a general LTI operator $p(D)$. The result is the

**Exponential Shift Law:**

(3) \[ p(D)(e^{rt}u) = e^{rt}p(D + rI)u \]

The effect is that we have pulled the exponential outside the differential operator, at the expense of changing the operator in a specified way.
14.2. **Product signals.** We can exploit this effect to solve equations of the form

\[ p(D)x = e^{rt}q(t), \]

by a version of the method of variation of parameter: write \( x = e^{rt}u \), apply \( p(D) \), use \( (3) \) to pull the exponential out to the left of the operator, and then cancel the exponential from both sides. The result is

\[ p(D + rI)u = q(t), \]

a new LTI ODE for the function \( u \), one from which the exponential factor has been eliminated.

**Example 14.2.1.** Find a particular solution to \( \ddot{x} + \dot{x} + x = t^2 e^{3t} \).

With \( p(s) = s^2 + s + 1 \) and \( x = e^{3t}u \), we have

\[ \ddot{x} + \dot{x} + x = p(D)x = p(D)(e^{3t}u) = e^{3t}p(D + 3I)u. \]

Set this equal to \( t^2 e^{3t} \) and cancel the exponential, to find

\[ p(D + 3I)u = t^2 \]

This is a good target for the method of undetermined coefficients (Section [11]). The first step is to compute

\[ p(s + 3) = (s + 3)^2 + (s + 3) + 1 = s^2 + 7s + 13, \]

so we have \( \ddot{u} + 7\dot{u} + 13u = t^2 \). There is a solution of the form \( u_p = at^2 + bt + c \), and we find it is

\[ u_p = (1/13)t^2 - (14/13^2)t + (85/13^3). \]

Thus a particular solution for the original problem is

\[ x_p = e^{3t}((1/13)t^2 - (14/13^2)t + (85/13^3)). \]

**Example 14.2.2.** Find a particular solution to \( \dot{x} + x = te^{-t} \sin t \).

The signal is the imaginary part of \( te^{(-1+i)t} \), so, following the method of Section [10] we consider the ODE

\[ \dot{z} + z = te^{(-1+i)t}. \]

If we can find a solution \( z_p \) for this, then \( x_p = \text{Im } z_p \) will be a solution to the original problem.

We will look for \( z \) of the form \( e^{(-1+i)t}u \). The Exponential Shift Law \( (3) \) with \( p(s) = s + 1 \) gives

\[
\dot{z} + z = (D + I)(e^{(-1+i)t}u) = e^{(-1+i)t}((D + (-1 + i)I) + I)u \\
= e^{(-1+i)t}(D + iI)u.
\]
When we set this equal to the right hand side we can cancel the exponential:

\[(D + iI)u = t\]

or \(\dot{u} + iu = t\). While this is now an ODE with complex coefficients, it’s easy to solve by the method of undetermined coefficients: there is a solution of the form \(u_p = at + b\). Computing the coefficients, \(u_p = -it + 1\); so \(z_p = e^{(-1+i)t}(-it + 1)\).

Finally, extract the imaginary part to obtain \(x_p\):

\[z_p = e^{-t}(\cos t + i \sin t)(-it + 1)\]

has imaginary part

\[x_p = e^{-t}(-t \cos t + \sin t).

14.3. **Summary.** The work of this section and the previous two can be summarized as follows: Among the responses by an LTI system to a signal which is polynomial times exponential (or a linear combination of such) there is always one which is again a linear combination of functions which are polynomial times exponential. By the magic of the complex exponential, sinusoidal factors are included in this.