13. Time invariance

As we have seen, systems can be represented by differential operators. A system, or a differential operator, is **time invariant** if it doesn’t change over time. A general $n$-th order differential operator has the form

$$L = a_n(t)D^n + \cdots + a_1(t)D + a_0(t)I$$

where each coefficient may depend upon $t$. It is time invariant precisely when all the coefficients are constant. In that case we have a characteristic polynomial $p(s)$, and $L = p(D)$.

The abbreviation LTI refers to the combination of the properties of linearity—that is, obeying the principle of superposition—and time invariance. These two properties in combination are very powerful. In this section we will investigate two implications of the LTI condition.

13.1. Differentiating input and output signals. A basic rule of differentiation is that if $c$ is constant then $\frac{d}{dt}(cu) = c\frac{du}{dt}$; that is, $D(cu) = cDu$.

The time invariance of $p(D)$ implies that as operators

$$Dp(D) = p(D)D.$$  

We can see this directly, using $D(cu) = cDu$:

$$D(a_nD^n + \cdots + a_0I) = a_nD^{n+1} + \cdots + a_0D = (a_nD^n + \cdots + a_0I)D.$$  

In fact the converse holds also; (2) is equivalent to time invariance.

**Example.** Suppose we know that $x(t)$ is a solution of the equation $Lx = 2\frac{d^4x}{dt^4} + 3\dot{x} + 4x = 2\cos t$. (I would not want to try to find $x(t)$ explicitly, though it an be done by the methods described earlier.)

Problem: Write down a solution of $Ly = \sin t$ in terms of $x$.

Well, up to multiplying by a constant $\sin t$ is the derivative of the right hand side of the original equation. So try $y = Dx$: $LDx = DLx = D(2\cos t) = -2\sin t$. By linearity, we can get to the right place by multiplying by $\frac{1}{2}$: we can take $y = -\frac{1}{2}Dx = -\frac{1}{2}\dot{x}$.

13.2. Time-shifting. Let $a$ be a constant and $f(t)$ a function. Define a new function $f_a(t)$ by shifting the graph of $f(t)$ to the right by $a$ units:

$$f_a(t) = f(t - a)$$
For example, \( \sin_{\pi}(t) = \cos(t) \). In terms of the language of signals, the signal \( f_a(t) \) is just \( f(t) \) but **delayed** by \( a \) time units.

Here is the meaning of time invariance:

If a system doesn’t change with time, then the system response to a signal which has been delayed by \( a \) seconds is just the \( a \)-second delay of the system response to the original signal.

In terms of operators, we can say: for an LTI operator \( L \),

\[
(Lx)_a = L(x_a)
\]

**Example.** Let’s solve the previous example using this principle. We have \( \sin t = \cos(t - \pi/2) \), so we can take \( y = \frac{1}{2}x(t - \pi/2) \).

Can you reconcile the two expressions we now have for \( y \)?
This section explains a method by which an LTI equation with input signal of the form $e^{rt}q(t)$ can be replaced by a simpler equation in which the input signal is just $q(t)$.

14.1. **Exponential shift.** The calculation \( \text{(1)} \)

\[
p(D)e^{rt} = p(r)e^{rt}
\]

extends to a formula for the effect of the operator $p(D)$ on a product of the form $e^{rt}u$, where $u$ is a general function. This is useful in solving $p(D)x = f(t)$ when the input signal is of the form $f(t) = e^{rt}q(t)$.

The formula arises from the product rule for differentiation, which can be written in terms of operators as

\[
D(vu) = vDu + (Dv)u.
\]

If we take $v = e^{rt}$ this becomes

\[
D(e^{rt}u) = e^{rt}Du + re^{rt}u = e^{rt}(Du + ru).
\]

Using the notation $I$ for the identity operator, we can write this as

\[
D(e^{rt}u) = e^{rt}(D + rI)u. \tag{2}
\]

If we apply $D$ to this equation again,

\[
D^2(e^{rt}u) = D(e^{rt}(D + rI)u) = e^{rt}(D + rI)^2u,
\]

where in the second step we have applied \( \text{(2)} \) with $u$ replaced by $(D + rI)u$. This generalizes to

\[
D^k(e^{rt}u) = e^{rt}(D + rI)^k u.
\]

The final step is to take a linear combination of $D^k$’s, to form a general LTI operator $p(D)$. The result is the

**Exponential Shift Law:**

\[
p(D)(e^{rt}u) = e^{rt}p(D + rI)u. \tag{3}
\]

The effect is that we have pulled the exponential outside the differential operator, at the expense of changing the operator in a specified way.
14.2. Product signals. We can exploit this effect to solve equations of the form

\[ p(D)x = e^{rt}q(t), \]

by a version of the method of variation of parameter: write \( x = e^{rt}u \), apply \( p(D) \), use (3) to pull the exponential out to the left of the operator, and then cancel the exponential from both sides. The result is

\[ p(D + rI)u = q(t), \]

a new LTI ODE for the function \( u \), one from which the exponential factor has been eliminated.

**Example 14.2.1.** Find a particular solution to \( \ddot{x} + \dot{x} + x = t^2e^{3t} \).

With \( p(s) = s^2 + s + 1 \) and \( x = e^{3t}u \), we have

\[ \ddot{x} + \dot{x} + x = p(D)x = p(D)(e^{3t}u) = e^{3t}p(D + 3I)u. \]

Set this equal to \( t^2e^{3t} \) and cancel the exponential, to find

\[ p(D + 3I)u = t^2 \]

This is a good target for the method of undetermined coefficients (Section 11). The first step is to compute

\[ p(s + 3) = (s + 3)^2 + (s + 3) + 1 = s^2 + 7s + 13, \]

so we have \( \ddot{u} + 7\dot{u} + 13u = t^2 \). There is a solution of the form \( u_p = at^2 + bt + c \), and we find it is

\[ u_p = (1/13)t^2 - (14/13^2)t + (85/13^3). \]

Thus a particular solution for the original problem is

\[ x_p = e^{3t}((1/13)t^2 - (14/13^2)t + (85/13^3)). \]

**Example 14.2.2.** Find a particular solution to \( \dot{x} + x = te^{-t}\sin t \).

The signal is the imaginary part of \( te^{(-1+i)t} \), so, following the method of Section 10, we consider the ODE

\[ \dot{z} + z = te^{(-1+i)t}. \]

If we can find a solution \( z_p \) for this, then \( x_p = \text{Im } z_p \) will be a solution to the original problem.

We will look for \( z \) of the form \( e^{(-1+i)t}u \). The Exponential Shift Law (3) with \( p(s) = s + 1 \) gives

\[ \dot{z} + z = (D + I)(e^{(-1+i)t}u) = e^{(-1+i)t}((D + (-1 + i)I) + I)u \]

\[ = e^{(-1+i)t}(D + iI)u. \]
When we set this equal to the right hand side we can cancel the exponential:

\[(D + iI)u = t\]

or \(\dot{u} + iu = t\). While this is now an ODE with complex coefficients, it’s easy to solve by the method of undetermined coefficients: there is a solution of the form \(u_p = at + b\). Computing the coefficients, \(u_p = -it + 1\); so \(z_p = e^{(-1+i)t}(-it + 1)\).

Finally, extract the imaginary part to obtain \(x_p\):

\[z_p = e^{-t}(\cos t + i \sin t)(-it + 1)\]

has imaginary part

\[x_p = e^{-t}(-t \cos t + \sin t).\]

14.3. **Summary.** The work of this section and the previous two can be summarized as follows: Among the responses by an LTI system to a signal which is polynomial times exponential (or a linear combination of such) there is always one which is again a linear combination of functions which are polynomial times exponential. By the magic of the complex exponential, sinusoidal factors are included in this.