18.03 Difference Equations and Z-Transforms

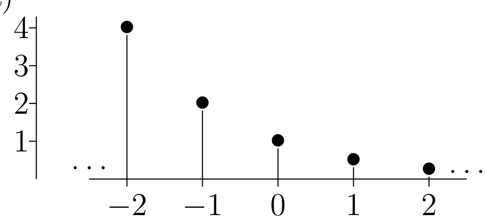
Difference equations are analogous to 18.03, but without calculus. On the last page is a summary listing the main ideas and giving the familiar 18.03 analogue. The one line summary is:

In 18.03 the answer is e^{at}, and for difference equations the answer is a^n.

Sequences $x[n]$ (also called signals or discrete functions)

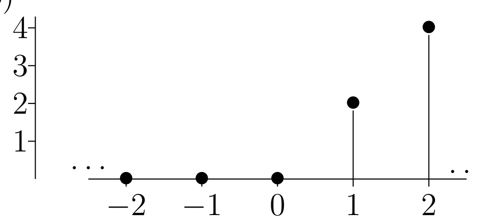
Examples:

a)



$x[n] = (1/2)^n$
$\Rightarrow \ldots x[-2] = 4,\ x[-1] = 2,$
$x[0] = 1,\ x[1] = 1/2,\ \ldots$

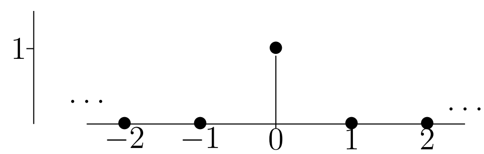
b)



$$x[n] = \begin{cases} 0 & \text{if } n < 0 \\ 2n & \text{if } n \geq 0 \end{cases}$$

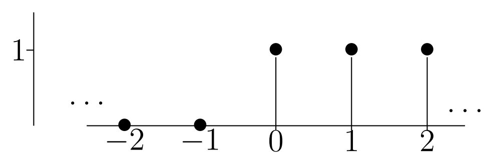
c) **Unit sample:**

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$



d) **Unit step:**

$$u[n] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geq 0 \end{cases}$$



Z**-transform** (analog of Laplace transform)

Let $x[n]$ be a sequence. Its z-transform is $X(z) = \sum_n x[n] z^{-n}$.

When it's useful we will denote the z-transform of x by $\mathcal{Z}x$ (similar to using $\mathcal{L}x$ for Laplace).

Example 1: z-transform of $\delta[n]$ is 1.

Example 2: z-transform of $u[n]$ is $U(z) = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + \ldots = \frac{1}{1 - z^{-1}}$.

Example 3: If $x[n] = 0$ for $n < 0$ and $x[n] = a^n$ for $n \geq 0$ then $X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}$.

(continued)

Convolution

Start with two sequences $x[n]$ and $y[n]$ their convolution is

$$(x * y)[n] = \sum_k x[k]\, y[n-k].$$

This arises in the following way. $X(z) = \sum_k x[k]z^{-k}$, $Y(z) = \sum_m y[m]z^{-m}$.

$$\Rightarrow\ X(z)Y(z) = \left(\sum_k x[k]z^{-k}\right)\left(\sum_m y[m]z^{-m}\right) = \sum_k \sum_m x[k]y[m]z^{-k-m} = \sum_n \left(\sum_k x[k]y[n-k]\right) z^{-n}$$

i.e., $X(z)Y(z) = \mathcal{Z}(x * y)$.

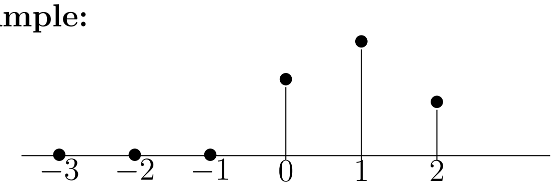
Operators

Delay operator: $(\mathcal{R}y)[n] = y[n-1]$. (Also called *right shift operator.*)

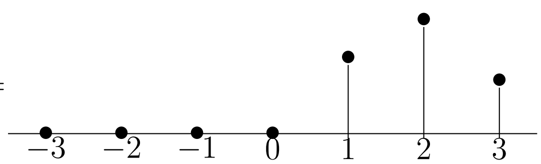
So, $(\mathcal{R}^2 y)[n] = y[n-2]$ etc.

Example:

$y =$



$\mathcal{R}y =$



Theorem: **($\mathcal{Z}$-transform of $\mathcal{R}$)** $\mathcal{Z}(\mathcal{R}x) = z^{-1}X$.

Proof: $\mathcal{Z}(\mathcal{R}x) = \sum_n (\mathcal{R}x)[n]z^{-n} = \sum_n x[n-1]z^{-n}$

$$= z^{-1}\sum_n x[n-1]z^{-(n-1)} = z^{-1}\sum_m x[m]z^{-(m)} = z^{-1}X(z). \quad \text{QED}$$

Difference Equations

Example 1: $y[n] - y[n-1] = x[n]$. ($x =$ input, $y =$ output)

Example 2: $y[n] + 8y[n-1] + 7y[n-2] = x[n]$.

Example 3: $y[n] + 8y[n-1] + 7y[n-2] = x[n] - x[n-1]$.

Example 4: $y[n] - ny[n-1] = x[n]$

Examples 1-3 are constant coefficient equations, i.e. linear time invariant (LTI). Example 4 is not constant coefficient. We will focus on constant coefficient equations.

(continued)

In practice it's easy to compute as many terms of the output as you want: the difference equation is the algorithm.

Example: For the difference equation $y[n] - \frac{1}{2}y[n-1] = u[n]$ find $y[n]$ for $n \geq 0$. Assume rest IC $y[-1] = 0$.

(Here $u[n]$ is the unit step function.)

answer: Rewrite the equation as $y[n] = u[n] + \frac{1}{2}y[n-1]$.

Make a table:

n	-1	0	1	2	3	4	...
$u[n]$	0	1	1	1	1	1	...
$y[n]$	0	1	3/2	7/4	15/8	31/16	...

General constant coefficient difference equations and the z-transform.

General form: $P(\mathcal{R})y = Q(\mathcal{R})x$. z-transform $P(z^{-1})Y = Q(z^{-1})X$.

x is called the input and y is the output or response.

Example: Example 3 above has $(1 + 8\mathcal{R} + 7\mathcal{R}^2)y = (1 - \mathcal{R})x$

Using the formula for the z-transform of $\mathcal{R}$ we get $(1+8z^{-1}+7z^{-2})Y = (1-z^{-1})X$.

System (or transfer) function

We call $\dfrac{Q(z^{-1})}{P(z^{-1})}$ the *system function*. Often, we will denote it $H(z)$.

Suppose we have a difference equation with *rest initial conditions*:

$P(\mathcal{R})y = Q(\mathcal{R})x$; $y[n] = 0$ for $n < 0$ then $Y = \dfrac{Q(z^{-1})}{P(z^{-1})}X$.

Thus, with rest IC, $Y(z) = X(z)H(z)$.

(The need for rest IC is explained in the odds and ends section later in these notes.)

Unit sample response

The unit sample response satisfies the equation $P(\mathcal{R})h = Q(\mathcal{R})\delta$ with rest IC.

It's easy to see that $\mathcal{Z}(h) = H$ is the system function.

Theorem: The equation $P(\mathcal{R})y = Q(\mathcal{R})x$ with rest IC has solution $y = x * h$, where h is the unit sample response of the system.

Proof: From above we know $Y(z) = X(z)H(z) = \mathcal{Z}(x * h)$. QED

Example: Solve $y[n] - ay[n-1] = \delta[n]$, with rest IC.

Z-transform: $(1 - az^{-1})Y = 1 \;\Rightarrow\; Y = \dfrac{1}{1 - az^{-1}} = 1 + az^{-1} + (az^{-1})^2 + \ldots$.

$$\Rightarrow\; y[n] = \begin{cases} 0 & \text{for } n < 0 \\ a^n & \text{for } n \geq 0 \end{cases}$$

(continued)

Poles, stability and homogeneous equations

The system $P(\mathcal{R})y = Q(\mathcal{R})x$ has system function $H(z) = \frac{Q(z^{-1})}{P(z^{-1})}$.

So the poles of $H(z)$ are exactly the roots of $P(z^{-1})$.

(We need to assume P and Q have no common factors.)

As in differential equations these poles give us the solutions to the *corresponding homogeneous equation*, i.e., $P(\mathcal{R})y = 0$.

Example: Solve the homogeneous equation $P(\mathcal{R})y = y[n]+8y[n-1]+7y[n-2] = 0$.

Trial solution: $y[n] = a^n$.

Substitution: $a^n + 8a^{n-1} + 7a^{n-2} = 0 \Rightarrow a^{n-2}(a^2 + 8a + 7) = 0$.

Characteristic equation: $a^2 + 8a + 7 = 0$.

Roots: $a = -7, -1$.

Two solutions: $y_1[n] = (-7)^n$, $y_2[n] = (-1)^n$.

General solution: $y = c_1 y_1 + c_2 y_2$. (This follows from the linearity of $P(\mathcal{R})$.)

(Below we will discuss the existence and uniqueness theorem that guarantees this gives all possible solutions.)

Note, $\dfrac{1}{P(z^{-1})} = \dfrac{1}{1 + 8z^{-1} + 7z^{-2}} = \dfrac{z^2}{z^2 + 8z + 7}$. So the roots of the characteristic equation are the same as the zeros of the denominator which are the poles of the system function.

Stability

As in 18.03, we say the system $P(\mathcal{R})y = Q(\mathcal{R})x$ is stable if the homogeneous solution $y_h[n] \to 0$ as $n \to \infty$ no matter what the initial conditions.

That is, the initial conditions don't affect the long-term behavior of the system.

Theorem: The system $P(\mathcal{R})y = Q(\mathcal{R})x$ is stable if and only if all the poles of the system function have magnitude < 1. (We assume P and Q have no common factors.)

Proof: As in the example just above, the general homogenous solution is a linear combination of sequences of the form $y_j[n] = a_j^n$, where a_j is a pole of the system function. This goes to 0 if and only if $|a_j| < 1$.

Graphically the system is stable if all the poles are inside the unit circle. (Compare this with differential equations where the homogeneous solution is built from functions of the form $y_j(t) = e^{at}$, so we need a in the left half-plane.)

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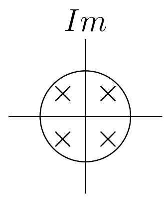
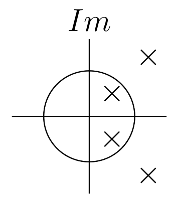
Example:

Difference equations:

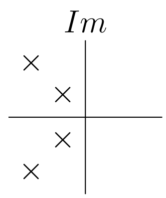


Stable discrete system

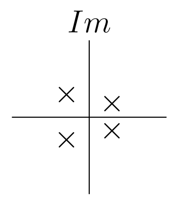


Unstable discrete system

Differential equations:



Stable continuous system



Unstable continuous system

Odds and ends

Causality: Causality means the future doesn't affect the past. Our systems $P(\mathcal{R})y = Q(\mathcal{R})x$ are causal because $y[n]$ depends only on $y[k]$ and $x[k]$ for $k \le n$.

Linearity: $P(\mathcal{R})$ is linear, i.e. $P(\mathcal{R})(c_1 y_1 + c_2 y_2) = c_1 P(\mathcal{R}) y_1 + c_2 P(\mathcal{R}) y_2$.
Proof: Immediate from the definition of $\mathcal{R}$ and $P(\mathcal{R})$.

Using linearity we see that the general solution to $P(\mathcal{R})y = Q(\mathcal{R})x$ is given by $y = y_p + y_h$, where y_p is any particular solution and y_h is the general homogeneous solution.

Example: Find the general solution to $y[n] - \frac{1}{2}y[n-1] = u[n]$.

Homogeneous solution: One pole at $z = \frac{1}{2} \Rightarrow y_h[n] = c\left(\frac{1}{2}\right)^n$.

Particular solution: Use rest IC, so $y_{[}n] = 0$ for $n < 0$.
We could find $y[n]$ for $n \ge 0$ directly, instead we'll find it using the z-transform.

$$(1 - \frac{1}{2}z^{-1})Y = \frac{1}{1 - z^{-1}} \Rightarrow Y = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} = \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 - z^{-1}}.$$

Using coverup we find $A = -1, \quad B = 2 \Rightarrow y_p[n] = \begin{cases} 0 & \text{for } n < 0 \\ 2 - (1/2)^n & \text{for } n \ge 0 \end{cases}$

Note, for $n \ge 0$ we can write the answer in the form $y_p[n] = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^n$.

Finally, the general solution is $y = y_p + y_h$.

Transient: If a system is stable then $y_h[n] \to 0$ for all initial conditions. In this case we call y_h the transient.

(continued)

First difference: One way to *discretize* a differential operator (using backward differences) is to replace D by the operator $\Delta_h = \dfrac{1-\mathcal{R}}{h}$. That is $\Delta_h y[n] = \dfrac{y[n]-y[n-1]}{h}$.
Let t be the continuous parameter, $y_c(t)$ a function and $y_d[n]$ the discrete approximation. Then $P(D)y_c(nh) \approx P(\Delta_h)y_d[n]$

Analogy with differential equations

Most often in 18.03 our system was $P(D)y = f$, but sometimes we had $P(D)y = Q(D)f$.
For example, for LRC circuits we have differential equations in q (the charge on the capacitor) and i (the current in the circuit). The input voltage E is handled slightly differently in each equation.

1. $Lq'' + Rq' + \frac{1}{C}q = E$. Input $= E$ is used as is.
2. $Li'' + Ri' + \frac{1}{C}i = E'$. Input $= E$ is modified.

In (1) we have $P(D)q = E$ and in (2) we have $P(D)i = Q(D)E$.

Theorem: (Existence and uniqueness) If $P(\mathcal{R})$ has degree m then the IVP

$$P(\mathcal{R})y = 0; \quad y[0] = b_0,\, y[1] = b_1,\, \ldots,\, y[m-1] = b_{n-1}$$

has a unique solution.
Proof: This is clear. Simply solve for $y[n]$ recursively as we did in the first order example. We show a degree two example.

Example: Solve $y[n] + a_1 y[n-1] + a_2 y[n-2] = 0, \quad y[0] = b_1, \quad y[1] = b_1$.
General equation: $y[n] = -a_1 y[n-1] - a_2 y[n-2]$.
$y[0] = b_0,\, y[1] = b_1 \;\Rightarrow\; y[2] = -a_1 b_0 - a_2 b_2 \;\Rightarrow\; y[3] = -a_1 y[2] - a_2 y[1]$, etc.
We see that $y[n]$ is uniquely determined.

Convolution formula as a result of linear time invariance

Consider the equation $P(\mathcal{R})y[n] = Q(\mathcal{R})x[n]$ with rest IC. Let h be the unit sample response. We will rederive the formula $y = x * h$ using linearity and time invariance.
Let $y[n] = (x*h)[n] = \sum_k x[k]h[n-k]$.

The sequence $h[n]$ is the solution to the equation $P(\mathcal{R})y[n] = Q(\mathcal{R})\delta[n]$.
Time invariance means that $h[n-k]$ is a solution to $P(\mathcal{R})y[n] = Q(\mathcal{R})\delta[n-k]$.
We can write $x[n] = \sum_k x[k]\delta[n-k]$, so by linearity we have

$$\begin{aligned} P(\mathcal{R})y[n] &= P(\mathcal{R})\sum_k x[k]h[n-k] = \sum_k x[k]P(\mathcal{R})h[n-k] = \sum_k x[k]Q(\mathcal{R})\delta[n-k] \\ &= Q(\mathcal{R})\sum_k x[k]\delta[n-k] = Q(\mathcal{R})x[n] \end{aligned}$$

We have shown that $y = x * h$ is a solution.

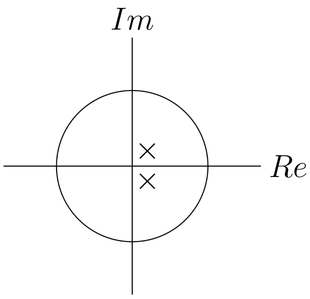
(continued)

Growth and decay rates

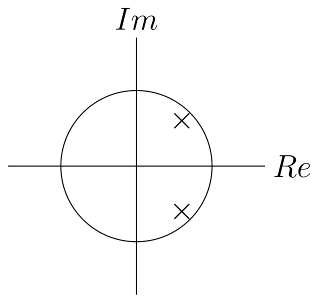
If a is a complex number then if $|a| < 1$ the rate that a^n *decays* to 0 depends on $|a|$, the closer to 1 the slower a^n decays. Likewise if $|a| > 1$ the rate that a^n *grows* depends on $|a|$.

If $a_1, a_2, \ldots, a_m$ are complex numbers then the growth or decay rate of the linear combination $y[n] = \sum c_j a_j^n$ is given by the biggest value of $|a_j|$. If all $|a_j| < 1$ then it is a decay rate and the bigger the rate (the closer to 1) the slower $y[n]$ decays.

Example: Both systems are stable. System A has a faster decay rate than system B, i.e., the transient disappears faster for system A than for system B.



System A



System B

The need for rest IC: We needed rest IC to write $Y(z) = \dfrac{P(z^{-1})}{Q(z^{-1})}$. We'll explain this using a simple example.

Consider $P(\mathcal{R})y[n] = y[n] - y[n-1] = \delta[n]$.

Particular solution with rest IC: $y_p[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$.

Homogeneous solution: $y_h[n] = c$.

General solution: $y[n] = y_p[n] + y_h[n] = \begin{cases} c & \text{for } n < 0 \\ 1 + c & \text{for } n \geq 0 \end{cases}$.

Since $P(\mathcal{R})(y_p + y_h) = \delta$ the z-transform gives $P(z^{-1})(Y_p + Y_h) = 1$. The reason we can't simply divide by $P(z^{-1})$ is because $P(z^{-1})Y_h(z) = (1 - z^{-1}) \sum_{n=-\infty}^{\infty} z^{-n} = 0$.

Algebraically we say that $P(z^{-1})$ and $Y_h(z)$ are *zero divisors*, that is, they are non-zero but when multiplied together they give 0. Just like dividing by 0, we have to be careful doing division with zero divisors. By demanding rest IC we make sure there are none of them around.

(continued)

Summary

Difference equations	*Differential equations*
Sequences: $x[n]$	Functions: $x(t)$
z-transform: $\mathcal{Z}(x) = X(z) = \sum_n x[n]z^{-n}$	Laplace transform: $\mathcal{L}(x) = X(s) = \int_0^\infty x(t)e^{-st}\,dt$
Convolution: $(x * y)[n] = \sum_k x[k]y[n-k]$	$(x * y)(t) = \int_0^t x(u)y(t-u)\,du$
$\mathcal{Z}(x * y)(z) = X(z)Y(z)$	$\mathcal{L}(x * y) = X(s)Y(s)$
Operator $\mathcal{R}$ = delay = right shift	D = derivative
$(\mathcal{R}x)[n] = x[n-1]$	$Dx = x'$

(The correspondence $\mathcal{R} \leftrightarrow D$ is an algebraic one. Look at the section labeled *first difference* to see an analytic correspondence.)

Causal LTI system: $P(\mathcal{R})y = Q(\mathcal{R})x$	$P(D)y = Q(D)x$ (In 18.03 most often: $P(D)y = x$)

(x = input, y = response, assume P and Q have no common factors.)

System function: $H(z) = \dfrac{Q(z^{-1})}{P(z^{-1})}$	$H(s) = \dfrac{Q(s)}{P(s)}$

(In 6.01 they also write $\dfrac{Q(\mathcal{R})}{P(\mathcal{R})}$ for the system function.)

Unit sample resp.: $P(\mathcal{R})h = Q(\mathcal{R})\delta$, rest IC	Unit impulse resp.: $P(D)h = Q(D)\delta$, rest IC
$\mathcal{Z}(h) = H$	$\mathcal{L}(h) = H$
$P(\mathcal{R})y = Q(\mathcal{R})x$; rest IC $\Rightarrow\ Y = XH,\quad y = x * h$	$P(D)y = Q(D)x$; rest IC $\Rightarrow\ Y = XH,\quad y = x * h$
Stability: poles of H inside unit circle	poles of H in left half-plane
Decay rate of transient determined by pole with greatest magnitude	decay rate determined by right most pole (greatest real part)