18.03 Muddy Cards 5: Mon, May 18, 2009

1. Can you post a detailed description of how everything fits together and how we moved from specific limited cases to more general systems?

Answer: We started with $x'(t) = f(t, x)$ the general nonlinear first-order equation. The linear case means

$$x'(t) = a(t)x(t)$$

in other words, linear in the dependent variable $x(t)$ not the independent variable t. We could not solve the nonlinear equation by a formula except if we got lucky. In the linear case there is a formula for the solution because the equation is separable.

I'm going to do the rest in more or less reverse order, so you see it from a different viewpoint. A general system has the form

$$\boldsymbol{x}'(t) = f(t, \boldsymbol{x}(t))$$

We only studied the autonomous case when $f(t, \boldsymbol{x}(t)) = g(\boldsymbol{x}(t))$, independent of t. This means the behavior of the system is independent of the time we start. If we run an experiment today or tomorrow what will matter is just the initial conditions and the time elapsed, not the day we ran the experiment. Moreover, we looked mostly at the 2×2 case.

In order to understand the 2×2 case, we specialized to the case of linear equations

$$x'(t) = ax(t) + by(t)$$
$$y'(t) = cx(t) + dy(t)$$

and moreover we required the coefficients a, b, c and d to be constant (independent of t), which means the system is also autonomous case. In this way the trajectories just depend on where in the (x, y)-plane you start, not when you start. We can draw them in the plane. We studied these trajectories systematically for all choices of a, b, c, and d except for some borderline cases (which we did understand as limits of the other cases). None of these is a special case of the others; there are just several different types.

In all the *nonborderline* cases the behavior of nonlinear systems resembles the behavior of linear ones at small scales. When the linearized system is at a borderline, the nearby nonlinear system can behave like any combination of the two sides and whatever fits in between those two behaviors (e.g. spiral sinks, spiral sources with ellipses or limit cycles in between).

Going backwards, a linear nth-order equation can always be turned into a $n \times n$ system using $x_1(t) = x(t)$ and $x_2(t) = x'(t)$, etc up to $x_n(t) = x^{(n-1)}(t)$ as dependent variables. All the information about systems can be translated back and tells us everything we previously knew about single equations (and covers more cases). We only discussed in any detail the case of 2nd-order equations

$$x''(t) + a(t)x'(t) + b(t)x = f(t)$$

In this case we use variables $x(t)$ and $y(t) = x'(t)$, and the single equation is replaced by the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ -ay - bx + f \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}$$

We only discussed in detail the case of constant a and b. In the constant case the so-called *companion matrix*

$$\begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$$

has the property that its characteristic polynomial is the same as the one for the original 2nd-order equation:

$$\det \begin{bmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{bmatrix} = \begin{vmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{vmatrix} = \lambda^2 + a\lambda + b$$

In this way, the 2nd-order equations are a special case of the systems, and one often keeps track of how a 2nd-order system behaves by looking at a 2-dimensional phase-plane plot of $(x(t)$ and $x'(t)$. When the characteristic polynomial has complex roots, the solutions are typically spirals (or in one borderline case, centers). If the spirals tend to the origin, then they are stable. If they tend to infinity then they are unstable. This corresponds to the phenomena we studied with 2nd-order equations: spirals are oscillating solutions and solutions that tend to zero are stable ones.

2. Also, I didn't understand your variation of parameters explanation. Please explain a little more.

Answer: Look at the notes for Lec 39, where there's lots (probably too much information by the end added in response to this muddy question). You can also look at the lecture in which this was first discussed.

3. Where do partial differential equations fit into (and beyond) the work we've done?

The main tool that is easy to get to right after this course is separation of variables, which reduces some basic (i.e. important) PDE to solving families of ordinary differential equations. This leads to the solution of many fundamental constant coefficient linear partial differential operators (heat/diffusion, wave, Laplace) of physics. It is discussed in EP 8.5–8.7 and Chapter 9. But to do it properly in higher dimensions requires familiarity with the Fourier transform (a bit easier in some ways than Fourier series, but involving improper integrals). Furthermore, a much more elaborate understanding of transforms is required to get to more detailed properties of solutions. For example, if your solution is expressed as a Fourier series or integral (or even worse and more general eigenfunction series) you may still have trouble summing the series and understanding what the function does. This is a long story. I will try to post a few further comments later after the course is over.

4. How does the wave equation fit into the context of 18.03? have we learned the necessary tools to solve it?

You have learned enough tools to solve it well in one space dimension. It's treated in the text in in 8.6. (Figure 8.6.1 is misleading. The whole point of the discussion is that it works only when θ is very small, which is not the case in the picture.)

As in the answer to Question 3 above, you don't have quite enough tools yet to solve it systematically in higher numbers of space dimensions. One especially useful tool is Fourier transforms, with which you can just as easily solve heat equations and Schrödinger's equation as the wave equation.