18.03 Muddy card Q and A. Spring 2009

Dates: Friday, April 3 and Friday, April 10.

Q 0. The solutions to Pset 7 were posted on the website before the deadline. Can you not do that again? It's really demotivating. (Similar note from one other student.)

A 0. I am sorry this happened. It will not happen again. Anyone who used those solutions violated the spirit of the course.

Q1. Can you please give a brief summary of integrating delta functions and step functions?

A 1. There is basically only one formula for integrating a delta function:

$$\int_A^B \delta(t-a)g(t)\,dt = g(a) \left(= \int_A^B \delta(a-t)g(t)\,dt \right)$$

provided $A < a < B$. The ambiguity of what to do when $a = A$ or $a = B$ is resolved only if we specify what we mean using $\pm$ on the limits. For example

$$\int_{0^-}^B \delta(t)g(t)\,dt = g(0)$$

Here are typical formulas in which delta functions appear. Example 1:

$$\int_{-\pi}^{\pi} \delta(t)\cos nt\,dt = \cos n0 = 1; \qquad \int_{-\pi}^{\pi^+} \delta(t-\pi)\cos nt\,dt = \cos n\pi = (-1)^n$$

Example 2:

$$\int_{0^-}^{\infty} \delta(t-a)e^{-st}\,dt = e^{-as} \quad (a \ge 0)$$

(equals 0 when $a < 0$).

For step functions, if $a \ge 0$ and $s > 0$, then

$$\int_0^{\infty} u(t-a)e^{-st}\,dt = \int_a^{\infty} e^{-st}\,dt = \frac{e^{-as}}{s}$$

Q 2. Pset 6 was long and very messy. We only seem to do derivations and theory in class, making application of these abstract concepts difficult on Psets. Every class on F. series is overloaded with formulas and π's without really understanding how it ties together. Help ?!

A 2. I hear your cry. There are only two main formulas in Fourier series, namely, the form of the series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

and the formula for the coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt; \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt.$$

There is a variant with L replacing π and $(\pi/L)nt$ replacing nt. We cannot get rid of the π's no matter how we rearrange the formulas.

The way Fourier series are used is that we replace each side of the differential equation by a Fourier series (finite or infinite sum). To differentiate a Fourier series, do it term by term. Same thing with integration and shifts and scale changes.

One other thing we do is to simplify formulas with the help of symmetries of sines and cosines. For example odd functions have sine series. Even functions have cosine series. Sines and cosines are even around their maxima and minima. They are odd around their zeros.

There is a larger framework of ways to represent functions by series, related to the scalar product of functions that I defined in lecture. That infinite-dimensional dot product idea is, indeed, an abstract, advanced topic.

Q 3. Were we expected to work on the Pset over spring break? It was fun, but it just took soooo long!

A 3. The problem sets are supposed to take about 7 hrs if they are for a regular week — longer if more than 3 lectures are covered. Evidently I missed the mark on Pset 6. You are supposed to work on the problems right after the relevant lecture, so there were a number of them that could be done during spring break.

Q 4. Why does the formula for Fourier series work to approximate functions? (Is there a quick proof?)

A 4. There is no quick proof. One upper level math course where this is proved is Fourier Analysis 18.103. Here is a rough description of how the proof goes (takes about an hour in an upper level class): There is a convolution formula

$$S_N(t) = \int_{-\pi}^{\pi} D_N(\tau) f(t-\tau) d\tau$$

for the partial sum of a Fourier series. The function $D_N(\tau)$ in this convolution integral is evaluated using a geometric series (with a complex ratio). Then there is some further cleverness, subtracting $f(t)$ from both sides and using a difference quotient, to see that the limit of the difference as $N \to \infty$ is 0. The proof I am describing works when f is differentiable or piecewise differentiable.

The fact that the Fourier series "usually" converges for continuous functions was proved in 1965 by a great Swedish mathematician named Lennart Carleson. He was was awarded the Abel Prize last year. The Abel prize is a recently inaugurated substitute for the Nobel Prize (which cannot, according to the terms of the will of Alfred Nobel, be awarded to a mathematician).

Q 5 and Q 6. How can we dot two trig functions as in $\sin nt \cdot \sin nt$? Aren't they scalars? Related question from April 10: I don't understand (a few lectures ago) how you knew to calculate $f \cdot g$.

A 5 and A 6. $\sin nt$ is a function, not a scalar. Here is a different way of thinking about what we did. Take a function $f(t)$ (for simplicity) on the unit interval $0 \le t \le 1$. Then consider the discrete locations $t_1 = 1/m$, $t_2 = 2/m$, $t_3 = 3/m$, …, $t_m = 1$. In place of $f(t)$ we can, for many practical purposes, such as approximate computations and plotting on a video screen, replace $f(t)$ by the list of values $f(t_1), f(t_2), f(t_3), \dots, f(t_m)$. We can regard this list as a vector of length m

$$\overrightarrow{A} = (a_1, a_2, \cdots, a_m), \quad \text{with } a_n = f(t_n), \quad (n = 1, 2, \dots, m).$$

Similarly, take another function g and let it correspond to the vector

$$\overrightarrow{B} = (b_1, b_2, \cdots, b_m), \quad \text{with } b_n = g(t_n) \quad (n = 1, 2, \dots, m).$$

Now instead of taking the usual dot product of the vectors $\overrightarrow{A}$ and $\overrightarrow{B}$, consider

$$\frac{1}{m}\overrightarrow{A}\cdot\overrightarrow{B} = \frac{1}{m}\sum_{n=1}^{m} a_n b_n = \frac{1}{m}\sum_{n=1}^{m} f(t_n)g(t_n)$$

This is exactly the same as the ordinary dot product, except for the normalizing factor $1/m$ in front. The advantage of the factor out front is that we can take the limit as $m \to \infty$,

$$\frac{1}{m}\sum_{n=1}^{m} f(t_n)g(t_n) = \sum_{n=1}^{m} f(t_n)g(t_n)\Delta t \ \to \ \int_0^1 f(t)g(t)\,dt \quad \text{(limit of Riemann sums)}$$

The difference between this and the dot product that we used is the interval (we used $-\pi < t < \pi$ and the normalizing factor, which was $1/\pi$ for us. As explained in the lecture notes, the factor $1/\pi$ was chosen so that $\sin nt \cdot \sin nt = 1$ (unit length) and similarly for cosines, except $\cos 0t$.

Q 7. How should we study for exams? The Psets do not reflect what the exams ask, unlike 18.01 and 18.02.

A 7. Look at the practice problems for the tests. Consult the exam review list (handout before the exam). The intention is to list the problem types you can expect on the exam. Go back and compare the list with what was actually on the exam. You should be able to recognize that most of these topics did appear on the exams we have had so far.

Q 8. In computing $F'(s) = \mathcal{L}(-tf) = -\mathcal{L}(tf)$ — in the last step, how did

$$\int_0^\infty f(t)te^{-st}\,dt$$

lead to the proof?

A 8. Note that by the definition of Laplace transform,

$$\int_0^\infty f(t)te^{-st}\,dt = \mathcal{L}(tf)$$

This fact is used in the third equality below

$$F'(s) = \int_0^\infty f(t)(-t)e^{-st}\,dt = -\int_0^\infty f(t)te^{-st}\,dt = -\mathcal{L}(tf)$$

(See also PS8 Part A problem on this; and the lecture notes.)

Q 9. Relationship between $\mathcal{L}(s)$ and $F(s)$? Why is there a table of fixed Laplace transforms, yet we should need to define $\mathcal{L}(s)$ specifically for each equation, i. e., should $\mathcal{L}(s)$ be different for each equation?

A 9. I can't quite follow this question, so you may have to pose it again. I think you may be asking about issues of notation. The notational conventions of this subject require getting used to. First $\mathcal{L}(s)$ is not a legal expression. We write

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \quad \text{(also denoted } \mathcal{L}(f(t)) = \mathcal{L}(f)\text{)}$$

Every specific function $f(t)$ has its own Laplace transform $F(s)$. No other function has that same Laplace transform. In other words, each time $f(t)$ changes, $F(s)$ also changes.

For example, if in a given problem we have the notations $f(t) = t$ and $x(t) = e^t$, then the corresponding Laplace transforms are $F(s) = 1/s^2$ and $X(s) = 1/(s-1)$.

If I rewrite your question replacing the word "equation" with "function" then it becomes: Is $F(s)$ different for each different function $f(t)$? *Answer:* yes $F(s)$ is different for each different $f(t)$. This is a very important property that allows us to know for sure that $F(s)$ determines $f(t)$.

Q 10. Could you please post a table of Laplace transforms?

A 10. Yes! Done.

Q 11. I'm confused about how to manipulate convolution integrals. (Pset 7, problem 4)

A 11. Most of the manipulations were changes of variables in integrals. Did you look a the solutions and did it still not make sense? I think this is something one needs to do in person, so we see more specifically what the difficulty is.

Q 12. I don't understand Fourier series or convolution. Can you write out a sample problem for each topic and solve step by step with notes on the side saying why you are doing each step? Ex. for Lecture on 4/8: 3rd definition $aw'' + bw' + cw = \delta$, I don't understand how to solve for it and (question trails off at this point)

We will have to do a Fourier series example in review in lecture. For now let me describe the 3rd way to look at the solution to $aw'' + bw' + cw = \delta$.

Step 1. $w(t) = 0$ for $t < 0$ (this is part of the definition of the weight function — the unit impulse response starts from nothing at all before time 0).

Step 2. $aw'' + bw' + cw = 0$ for $t > 0$ (because δ is zero for $t > 0$)

Step 3. Step 2 implies $w(t) = c_1 h_1(t) + c_2 h_2(t)$ where h_1 and h_2 solve the homogenous equation of Step 2. For example, if $w'' + 4w = 0$ were the equation, then the form of the answer would be $w = c_1 \cos 2t + c_2 \sin 2t$.

Step 4. To specify the constants c_1 and c_2 we need to recognize that the δ function will come from a jump in w' across $t = 0$.

$$w''_{singular}(t) = (w'(0^+) - w'(0^-))\delta(t)$$

Since we want $aw''_{singular}(t) = \delta(t)$, we will want $(w'(0^+) - w'(0^-)) = 1/a$. But $w'(0^-) = 0$, so it must be that

$$w'(0^+) = 1/a$$

Finally, we don't want any jump in w across $t = 0$. If there were one then we would already get a δ function from w' and w'' would contain a derivative of a delta function (too singular for us). No jump in w across $t = 0$ translates into

$$w(0^+) - w(0^-) = 0$$

But $w(0^-) = 0$, so we also have $w(0^+) = 0$. In all, our two boundary conditions are

$$w(0^+) = 0; \quad w'(0^+) = 1/a$$

In any specific instance, we can plug in the conditions to evaluate c_1 and c_2. For example, in the case $w'' + 4w = \delta$ we have $w = c_1 \cos 2t + c_2 \sin 2t$ $(t > 0)$ and

$$w'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t, \quad t > 0$$

Therefore, $w(0^+) = c_1 = 0$ and $w'(0^+) = 2c_2 = 1$. So $c_2 = 1/2$ and

$$w(t) = (1/2)(\sin 2t)u(t)$$

Q 13. Why didn't we learn Laplace transform before learning convolution?

A 13. Convolution is a fundamental and basic idea that does not require Laplace transforms. Convolution comes about as the solution operator to a differential equation with initial conditions zero. The convolution also pops out of the Laplace transform which is not surprising considering that the Laplace transform can be used to solve initial value problem and so can convolution. It turns out that the Laplace transform transforms convolution of functions of t into multiplication of functions in the s variable! This means that convolution and the Laplace transform are closely linked. (Actually the full picture leads to something called the Fourier-Laplace transform.)

One could teach an entire course on differential equations without mentioning the Laplace transform, but it would be unacceptable to omit convolution. We discuss the Laplace transform because it is so frequently used in engineering disciplines and also in probability theory. But mathematicians and physicists, for instance, often prefer to analyze differential operators using Fourier series and Fourier transforms.

Q 14. Could you give a brief summary at the beginning of each class explaining the purpose of what we're doing? It would help me to understand the application of the concepts much better!

A 14. I would love to be clearer. What we are doing in half of unit 1, all of unit 2 and all of unit 3 is setting up various techniques to solve constant coefficient linear differential equations. The broader purpose of 18.03 is to increase the scope of techniques you know and your mathematical vocabulary. But you are probably not talking about broader purpose, but rather the goals of that individual lecture. What I try to do is to tell you the topic of the lecture, but I think you are suggesting that I tell you in the first few minutes of the lecture what to expect to get out of the lecture in the way of new techniques. Sometimes that is hard because the class is spent developing the notations to express what we are doing. Other times, I may indeed have blown an opportunity to be clearer by not pointing out the way we are going.