18.03 Muddy Q and A, Fri Feb 20

1. Could you post a list of all diff eq we should memorize? (and another questioner asks) Could you reexplain how to get x_p and x general solutions when solving $x' + kx = ae^{rt}$?

Keep in mind that the names of variables and parameters and presentation may change. Moreover, even having memorized a formula, you will also want to know how to derive it. In some cases, you may want to know more than one way.

a) One thing you must know is the integrating factor method for solving a linear equation

$$x'(t) + k(t)x(t) = b(t)$$

Memorize the procedure because you will be asked to carry it out. For now it is unwise to apply the general integral formula we derived using the integrating factor procedure (or the fundamental solution procedure). This formula is quite a bit harder to use in the cases we are considering than the integrating factor procedure.

b) The general form of the solution to a linear equation is

$$x(t) = x_p(t) + ch(t)$$

where y_p is any single solution to the equation (called a particular solution with a subscript p) and h solves the related, but simpler so-called homogeneous equation

$$h' + kh = 0$$

(same left side, but right side equal to zero). The fact that $x = x_p + ch$ solves the equation follows from the the fundamental property of linear equations concerning sums of inputs. Take x_p $x_p' + ax_p = b$ and $h' + kh = 0$, then adding we get $x = x_p + ch$ satisfies

$$x' + kx = (x_p + ch)' + k(x_p + ch) = x_p' + kx_p + c(h' + kh) = b + c \cdot 0 = b$$

(c constant, but $k = k(t)$ and $b = b(t)$ can be variable.

c) A more specific fact that I would memorize has to do with the form of the solution to

$$x' + kx = be^{rt}$$

when k and b are constants. Namely, the answer has the form

$$x = c_1e^{rt} + ce^{-kt}$$

The constant c can be anything. The constant c_1 is determined by plugging c_1e^{rt} into the equation and figuring out which constant c_1 works. This is known as the method of undetermined coefficients. (When you find c_1, at some point, you will divide by $r + k$. So this method goes wrong if $r = -k$. Don't worry we will address this in Unit 2. We can already find the solution when $r = -k$ using an integrating factor.)

You have to get practice with this formula by working several examples. We will derive it twice and mention a third instance.

i) Consider

$$x' + kx = b$$

that is, the case k and b are constant and $r = 0$. The answer is

$$x = b/k + ce^{-kt}$$

The integrating factor method works but is much too slow. Save it for more complicated equations in which the k term depends on t. Instead we use the **the method of undetermined coefficients**. A good guess for a particular solution is a constant $x = c$. Plugging in we get $0 + kc = b$, so $c = b/k$. Thus $x = b/k$ solves the equation.

Next, we find all the solutions to the homogeneous equation $h' + kh = 0$. This we do by remembering that a solution is $h = e^{-kx}$. The first five times we do it, however, we use the technique of separation:

$$h' = -kh \implies dh/h = -kdt \implies \ln h = -kt + c \implies h = Ae^{-kt}$$

Adding the particular solution $x = b/k$ to the homogeneous solution we get the general solution to $x' + kx = b$ for constant k and b, namely,

$$x = b/k + ce^{-kt}$$

ii) The same method works for $x' + kx = e^{rt}$. The guess for the particular solution is c_1e^{rt}. Plug in for x to get

$$c_1re^{rt} + kc_1e^{rt} = e^{rt} \implies c_1(r + k)e^{rt} = e^{rt} \implies c_1(r + k) = 1$$

Therefore,

$$c_1 = 1/(r + k)$$

and

$$x_p = \frac{e^{rt}}{r + k}$$

solves $x_p' + kx_p = e^{rt}$. Multiply through by a to get

$$w_p = ax_p = \frac{a}{r + k}e^{rt}$$

solves $w_p' + kw_p = ae^{rt}$ which is what the particular solution to the problem asked above. For the general solution add ch,

$$x = ax_p + ch = \frac{a}{r + k}e^{rt} + ce^{-kt}$$

is the general solution to $x' + kx = ae^{rt}$.

iii) Finally, the same method works for complex-valued functions in the equation

$$z' + kz = e^{i\omega t}$$

2. Consider $x' + kx = \cos t$. Why is it that the solution to this equation is the real part of the solution to

$$z' + kz = e^{it}$$

in which $z = x(t) + iy(t)$? (and) Explain the jump from $x' + 2x = \cos t$ to $z' + 2z = e^{it}$. Re-explain complex exponentials, linear response to sinusoidal models.

To answer the first question, write out the left side of the equation:

$$z' + kz = (x + iy)' + k(x + iy) = (x' + kx) + i(y' + ky)$$

On right side we have, $e^{it} = \cos t + i \sin t$. Therefore, if we take the real part of $z' + kz = e^{it}$ we see that

$$x' + kx = \mathrm{Re}\,(z' + kz) = \mathrm{Re}\, e^{it} = \cos t$$

The explanation and answer to the second question is that we have already done the computation once so that we know it works (as above). Similarly to what we just did

$$z' + 2z = e^{it}$$

with $z = x + iy$ is the same as

$$z' + 2z = (x + iy)' + 2(x + iy) = (x' + 2x) + i(y' + 2y) = \cos t + i \sin t \ (= e^{it})$$

Thus it contains the following two equations when split as above into real and imaginary parts:

$$x' + 2x = \cos t \quad \text{and} \quad y' + 2y = \sin t$$

If we solve for z then we will have found both $x(t)$ and $y(t)$.

For the last two questions, what is above should help. Also, to get to the amplitude/phase shift presentation, you should practice with polar form. For example, for amplitude, look at question 10 below. For conversion from rectangular to polar form, if $a > 0$ then

$$a + ib = re^{i\phi}$$

where $r = \sqrt{a^2 + b^2}$ and $\tan\phi = b/a$, $-\pi/2 < \phi < \pi/2$.

3. What does it mean to be linear versus homogeneous versus first-order versus separable? What is the significance?

So far we have only discussed first-order equations, namely the ones involving only the first derivative of the unknown, not the second or higher derivatives. Among first-order equations, there are special types that are solvable by systematic methods and formulas. The significance of recognizing these types is then that we can solve the equations. Among these types, the notion of linearity is the most important, both in this course and beyond. A very large part of analysis of differential equations is figuring out in what ranges of values one can replace the equation by a manageable linear differential equation.

The form of a first-order differential equation is $y' = f(x, y)$. The equation is called **separable** if f has the form $f(x, y) = F(x)G(y)$. If so we can solve by separating to $dy/G(y) = F(x)dx$ and integrating. The form of a **linear** equation (of first order) is $y' = a(x)y + b(x)$. It can be solved using an integrating factor (a method that can also be applied to linear equations with higher derivatives and to systems of equations).

A homogeneous differential equation (not to be confused with the notion with the same name in connection with linear equations) means an equation in which $y' = f(x, y)$ for a function with the symmetry $f(sx, sy) = f(x, y)$ for all $s > 0$. It turns out that this reduces to a separable equation for the variable $u = y/x$. Unlike the previous two cases (separable and linear), you are not asked to remember this. If I want to give you a problem with this symmetry I will remind you about the substitution. (Homogeneity is useful and comes up a great deal. Symmetry in it's higher-dimensional geometric forms has and even larger significance. Historically it has been the single most powerful theoretical tool in discovering new physical laws, including the standard model unifying three of the four forces of nature. A general principle of mathematics and physics is that equations with symmetries can be simplified by a suitable change of variable. For a first-order differential equation with only one unknown, a single symmetry suffices to make it possible to make a substitution and solve the problem completely. In general, each symmetry simplifies the problem by reducing the number of unknowns by one.)

4. Find y_p from a null cline. How do we know that the steady solution on a y' versus y graph occurred at the zero?

Finding a constant solution in the case $y' = f(y)$ works as follows. We look for a number y_0 such that $f(y_0) = 0$. Then the function $y(x) = y_0$ is constant so that $y'(x) = 0$. It also happens that $f(y(x)) = f(y_0) = 0$. So the equation $y' = f(y)$ is satisfied by this function. This can only work at the places were f is zero because the slope of a constant function is zero.

Note the important and special case of linear equations which came up on the problem sets in connection with steady state temperature, for instance. Find y_p from a 0 isocline if $y' = ay + b$ and a and b are constant. Solve $ay + b = 0$ to get $y = -b/a$. This constant satisfies $y' = 0$, so is a solution.

As for the word usage in these two questions, it turns out that the expression y_p (particular solution) is used customarily only in connection with linear equations. as for the y' versus y graph, this is a bit confusing way to refer to the graph of f.

5. Are there any "good" sources to understand this?

Depends on that "this" is. We have a textbook, supplementary notes, notes from lecture. The problem set is supposed to help (at a minimum to see what you don't yet understand). There are the practice exams and a few other review materials at out web site.

6. How do you get from $y = \frac{5}{3} + ce^{3t}$ to

$$y = \frac{5}{3} + e^{3(t-t_0)}$$

or

$$y = \frac{5}{3} + 0$$

or

$$y = \frac{5}{3} - e^{3(t-t_0)}$$

(from Lec 8).

Let us go backwards. The first case,

$$e^{3(t-t_0)} = e^{3t-3t_0} = e^{3t}e^{-3t_0} = Ae^{3t}$$

where $A = e^{-3t_0}$. If t_0 ranges over all possible real numbers, then A ranges over all positive numbers $A > 0$. This corresponds to $c = A > 0$. Similarly,

$$-e^{3(t-t_0)} = -Ae^{3t}$$

where $-A$ ranges over all negative numbers, corresponding to $c = -A < 0$. Finally, the middle case is $c = 0$.

7. Could you explain further the bifurcation diagram? (and) Phase line versus bifurcation diagram?

The phase line has points at each steady solution (whether stable or not). It also has arrows indicating whether y is increasing or decreasing in between places where $f(y) = 0$. Our bifurcation diagram is a phase line on each of it's vertical slices with the stable solutions marked (but the arrow information omitted). It could be reintroduced by choosing a color for $f > 0$ and another for $f < 0$.

Visit it yourself at the daimp web site. More generally, a bifurcation diagram is a way to keep track of long term behavior of differential equations. It is also used to analyze iterative computational schemes. The diagram plots persistent solutions (or limit values of solutions) to a differential equation as a function of some parameter, often a coefficient in the equation. Sometimes there are no persistent solutions or no limits (the solutions tend to $\pm\infty$). In that case there is no curve above the corresponding parameter value. Sometimes there is as single well-defined limit and the plot looks like an ordinary function. Finally, sometimes (and this is the interesting case called bifurcation) there are two or more distinct limits depending on initial conditions. The bifurcation diagram plots this as a multi-valued function (two or more curves above one point along the horizontal axis). What usually happens to a limiting behavior varies continuously with the parameter. s But when two limits meet they sometimes cancel each other out or merge into one solution (or split into more solutions). This is a curious, discontinuous sort of behavior. A typical example is a case in which the parameter is some kind of energy and the higher the energy the more solutions and limiting behaviors one sees. The solutions keep dividing and spawning new ones as energy increases.

8. $y = \dfrac{5}{3} + ce^{3t}$ Is the term ce^{3t} also called the transient term? (and) Can we define the terms homogeneous solution/particular solution and transient/persistent formally to distinguish between them?

The answer to the first question is no: ce^{3t} is not transient because it does not tend to zero. Transient means that the effect diminishes to nothing over time. A transient term by definition tends to zero (becomes negligible) as $t \to \infty$. By contrast, in the limit the term ce^{3t} is much more important than $5/3$.

This family of functions solves $y' = 3y - 5$. The solution $y = 5/3$ is a particular solution. On the other hand, any of the others, such as $5/3 + 22e^{3t}$ is also a particular solution. A particular solution is any single solution.

The homogeneous solutions are $h = ce^{3t}$. (Once you have found one particular solution we can produce all others adding multiples of a solution homogeneous solution, as above in 1b, where also the homogeneous solutions are defined.) The only steady or persistent or equilibrium solution is $5/3$. Because small perturbations change the long-term behavior, this is an unstable equilibrium, which can never be witnessed in an experiment.

9. Is there some way to actually solve the "system" part? What can we deduce from this form $\left(\dfrac{1}{k}\dfrac{d}{dt} + 1\right)$? Why is it better to change from $T' + kT = Y$ to

$$\left(\frac{1}{k}\frac{d}{dt} + 1\right) T = Y$$

Yes to the first question. There s a way to solve the "system" part. If we write

$$Q = \left(\frac{1}{k}\frac{d}{dt} + 1\right)$$

Then we will write down integral operations that invert this operation:

$$QT = Y \implies T = Q^{-1}Y$$

although we will have to be careful with initial conditions.

To address the third question I need to call attention to a misprint in it. What we started with was $T' + kT = kY$, not $T' + kT = Y$. The system (through the parameter k) appears on both sides of the equation. What we have gained when we divide by k is the simplicity of putting k in only one place instead of two and putting the effects of the system all on one side of the equation. When we do this, we see that the units match (T and Y have the same units in the discussion about heights of tides) Furthermore, k is matched with the time variable and a change of time kt in place of t will transform this to an equation without numbers, only absolute constants. There is a nature time scale to the problem related to k.

Similar ideas arise in all kinds of engineering, physics and math. For example, there is a natural length scale of fluids. You can compare water (a fluid) with air, which follows the same fluid mechanics equation. The difference is that air is much more viscous. But the comparison is more subtle than that. The two fluids look the same, depending on the lengths involved. If you imagine a fish that stops swimming, then it will come to rest in about one fish length. If you imagine an airplane gliding, then it will come to rest after several hundred plane-lengths. If you imagine a cell or an object of similar size that can propel itself, then if it stops "swimming" it will stop rather suddenly (in a tiny fraction of its length). Thus if we understand this relative, scaled measurement we can see exactly how much more viscous water is than air and at what scale we should be comparing phenomena across all fluids.

10. Binomial theorem. Why is it beneficial? Also, how is the modulus related to the polar form of an expression?

Last question first: $|re^{i\theta}| = r$. (This is very useful for us, letting us read off the amplitude of oscillatory functions without even computing the real part). It is proved by noting that

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

The binomial theorem says, for instance, $(a+b)^2 = a^2 + 2ab + b^2$. It also gives the formula for $(a+b)^3$, $(a+b)^4$, etc. It is just a basic fact in algebra. It gets more and more complicated as the powers increase and the coefficients (known as combinatorial coefficients) play a big role in combinatorics, a subject with lots of uses in math and computer science. Not very many uses here in this class. It came up in connection with formulas for $\sin n\theta$ in terms of $\sin\theta$ and $\cos\theta$, which in turn is connected to nth roots and lots of algebra.