18.03 Muddy cards: Week 1, Feb 6, 2009

Number of queries/topic

10	Context/Why? (in general and spread among topics below)
15	Runge-Kutta, RK4 (especially the vector diagram)
2	RK2 (Improved Euler)
4	RK1 (Euler)
5	RK accuracy
3	Pitfalls in numerics
6	Lobster trap (+ two after class)
4	Direction fields/isoclines
3	Existence/uniqueness/limitations
5	Write larger!

Questions and Answers

1. How are isoclines and the direction field connected? Why are we doing all this.

The direction field is a collection of little segments indicating the slope that the solution curves of $y' = f(x, y)$ must have (at whatever points we choose to draw them). The direction field consists of little pieces of what would be tangent lines to solution curves. We sketch them in order to get an idea what the solution curves will look like and how they behave. Once we have a direction field, we can draw solution curves (approximately) by making them tangent to each direction field segment they encounter. Since only a sampling of direction field segments is drawn, one has to extrapolate what the appropriate slope is.

The direction field is potentially a large, complicated mess. In order to draw a good sampling, it helps a lot to do something systematic. This is why we use isoclines. The isoclines are the level curves of $f(x, y) = m$, for each fixed constant m. In class we looked at the equation $y' = y^2 - x^2$ and one isocline was $y^2 - x^2 = 1$ which is the two branches of a hyperbola. Along this hyperbola, the slope is $m = 1$, so the little segments we draw along these hyperbolas all have slope 1 and are centered on the hyperbola(s).

2. Why do we choose $y' = 0$? Does this become an asymptote in most situations?

All isoclines are helpful to understand what solutions are doing. The first reason to use the null-cline $f(x, y) = 0$ is that 0 is the easiest number to work with. A second reason is that the cases $y' = 0$ are the critical points (maxima, minima) of the solutions y.

There is a situation in which the null-cline $f(x, y) = 0$ is an essential feature, namely, when $f = f(y)$, a function of y alone. A constant b satisfying $f(b) = 0$ has the property that $y = b$ (the constant function) solves the equation. For example, as we saw in class, if $y' = y^2 - 2y + 1$, then the null-cline $y^2 - 2y + 1 = 0$ corresponds to the horizontal line $y = 1$. If you draw the other isoclines, you can see that this is an asymptote, starting from any (x_0, y_0) with $y_0 < 1$. (When $y_0 > 1$, the solutions tend to infinity.) The constant solution $y = 1$ is called semistable because from one direction it is an asymptote, and from the other it is not.

In general, null-clines don't have to be asymptotes. The minute there is an x in the equation, you should not expect it. For example, in class we found the solutions to $y' = x - y$ are $y = x - 1 + ce^{-x}$. The asymptote is $y = x - 1$. In this example, there is still a coincidence, namely, it happens that an isocline $x - y = -1$ (slope -1) also happens to be a solution. You should not expect some isocline to also be a solution. (Note this only happens if there happens to be straight line that is a solution.)

3. How do you solve $y' = x + y$, $y(0) = 2$?

This is a linear equation, which we now know how to solve (in Lec 3 via an integrating factor) but did not know how to solve when the questioner wrote. He/she came up with it because it was used (with initial condition $y(0) = 1$) to illustrate numerical methods in Chapter 6. The solution was stated without explanation because by that stage of the book, solving linear equations is standard. This is going to be typical for you in higher level courses. Writers and lecturers are going to take for granted that you have a basic knowledge of lots of things. It's a good idea to recognize (as this questioner did) what you don't know and to pinpoint it.

4. In solving a differential equation, what exactly is the answer (formatting etc.)?

The standard answer is a formula. The general solution to $dy/dx = ky$, where k is a constant, is ae^{kx}. If we require an initial condition like $y(0) = 2$, then the answer is more specific, $y = 2e^{kx}$. On the other hand, in a numerical example, the answer might be a table of values or an image of the solution curve. What we want out of a differential equation is often not a formula for the answer, but some interpretation of what happens to the solution, such as what it does as $x \to \infty$.

5. Is there a way to derive RKn for arbitrary integer n? You said RK8 is "horrible." Does that mean low accuracy or high complexity?

Yes. RK4 is discussed in Wikipedia, along with the higher order methods and references. The higher order methods have higher accuracy but also higher complexity. The reason RK4 is used is that for functions $f(x, y)$ in "reasonable" size ranges and for accuracy up to, say, 15 digits, RK4 is generally faster. For a higher order method to pay off you would need to be investigating a problem that required more accuracy than this. If I had to guess, I would say the payoff would begin around 50 or 100 digits. Situations where we care about such accuracy are very rare. An off-the-shelf program would be wasting time to address them. Matlab keeps about 15 digits in the background and displays 4 digits after the decimal point.

6. I didn't follow the diagrams for RK4. RK4 pictures were confusing. How do you find Q_4 in RK4? (and many more)

As I said in lecture, I will not be spending enough time describing RK4 for you to figure it out completely. If you want the full story, look at EP Chap 6. The Matlab implementation is EP Figure 6.3.12.

Here is a review of the overall idea. You can reproduce the vector diagram yourself if you follow this paragraph closely. We use an average of four slopes at well-selected points in the vicinity of the step that is being taken. The four points are as follows. Q_1 is the starting place $Q_1 = P_n = (x_n, y_n)$. Q_2 is the place where a single step of the Euler method starting from Q_1 would land if the step size were $h/2$. Q_3 is where the Euler method starting from Q_1 would land if the step size were $h/2$ but using the slope from Q_2, not Q_1. Finally, Q_4 is the point we get from Euler's method starting from Q_1 as usual, but this time we take step size h and use the slope at Q_3. Sounds complicated? Well we are not done yet: the formula for the slope we use is the **weighted average** of the four slopes at Q_1, Q_2, Q_3 and

Q_4 (double weighting to the middle two Q_2 and Q_3 that use step size $h/2$) .

One way to get a feeling for RK1, 2, 4, is to consider the case $y' = f(x)$, in which f depends only on x not y. In that case, RK4 corresponds to Simpson's rule. (Similarly, Improved Euler corresponds to the trapezoidal rule and ordinary Euler's method to the ordinary Riemann sum with left endpoints.)

7. In Modified Euler what were those vectors with double arrows?

The different arrows (double arrows, etc) on vectors were labels to indicate similar vectors. In other words, the two vectors marked with double arrows were supposed to be parallel, one with base point Q_2 and the other with shifted base point Q_1.

8. Why do we use Euler's method? In what context? (more later in error estimates)

Numerical methods are how we solve differential equations when we can't solve them with formulas. Most differential equations do not have answers by formulas. Even when we do have a formula, we still have to get numbers out sometimes, which requires evaluating functions like e^x numerically. Moreover, if we are making an image we use a numerical formula. We discussed Euler's method in detail because it is the easiest. We can even implement it by hand (even on a test). RK4 is more practical for a computer but requires too much arithmetic for us to carry out by hand.

9. Where does hA_0 come from in Euler's method? What is the relationship to the table of Euler's method to the applet? What is n on the table?

A_0 is the slope of the curve at $(x_0, y_0) = P_0$. If we follow the tangent line with slope A_0 starting at (x_0, y_0) a horizontal distance h (leg of right triangle) then the vertical leg is hA_0, so that the next point is $x_1 = x_0 + h$ and $y_1 = y_0 + hA_0$. The vector pictured in the applet is the hypotenuse, the vector from $P_0 = (x_0, y_0)$ to $P_1 = (x_1, y_1)$. When you click "start" in the applet, that vector is drawn. When you click again, the next Euler step from P_1 to P_2 is drawn, etc. The number n in the table is a label keeping track of which step we are at. The nth step corresponds to $P_n = (x_n, y_n)$. In the table our aim is to find out successively the value of each y_n, $n = 0, 1, 2, \ldots$.

10. Explain the bounds for RK1, RK2, RK4, of Ch, Ch^2 and Ch^4, and pitfalls.

We discussed Euler's method to give you some experience with how far you can trust a numerical method to give the right answer. There are very rare instances when a scientist might actually prefer Euler's method to RK4, such as when the function $f(x, y)$ is polygonal shaped, not smooth. Then all the advantages of RK4 evaporate. This is because the C in the error estimates for RK1 = Euler depends only on first derivatives of $f(x, y)$, $\partial f/\partial x$ and $\partial f/\partial y$. The higher-order methods depend on higher derivatives.

As a practical matter, we start trusting the numerical result when the discrepancy between the estimate at step size $h = .1$ and $h - .01$ and $h = .001$ follow the correct power law, which you can diagnose by pretending that the smallest scale is the exact answer. On the other hand, to be certain, we need a more theoretical argument. (See extra credit problem, due with PS3 to be posted.)

To get a feeling for the accuracy levels of RK1, RK2, RK4, consider a simpler problem of estimating the value of a function using nearby values (at distance h). This is roughly what we are doing when we evaluate at those points Q_1, Q_2, etc. The simpler problem we will consider is how closely can you approximate a function $f(x)$ by $f(x \pm h)$. First of all,

$$|f(x + h) - f(x)| \leq C_1 h$$

where $C_1 = \max|f'|$ (proved using the fundamental theorem of calculus). If we use **both** $f(x+h)$ and $f(x-h)$ we can do much better:

$$|(f(x+h)+f(x-h))/2 - f(x)| \leq C_2 h^2$$

where $C_2 = \max|f''|$. Try it out for a few polynomials with $x = 0$ to see. (This can also be proved using the fundamental theorem.) It turns out that you can do the same with four points $f(x \pm h/2)$ and $f(x \pm h)$ but now you need to take the correct weighted average. You can capture $f(x)$ to within an error of size $C_4 h^4$ with a constant C_4 that depends on the fourth derivative of f.

11. What does the constant refer to when we solve a differential equation? Explain families of of curves, orthogonal families.

A function $y(x)$ is specified by its rate of change and one more constant, namely, at what value it starts. This is called a constant of integration. When we specify the rate of change of a function y by saying $y' = f(x, y)$, we have not specified the function completely, but only up to a constant that it not quite a constant of integration, but can still be nailed down by specifying a particular initial value $y(x_0) = y_0$. The general solution to a first-order differential equation will essentially always be a family of functions with one extra parameter or constant that can be evaluated if we pick an initial condition.

A family of curves satisfying one differential equation like $y' = x + y$ will fill up all of space with one curve through each point (by the existence and uniqueness theorem). The orthogonal family is the family of curves perpendicular to the first family. For example, concentric circles around the origin fill up space. The orthogonal family is the family of lines through the origin (or rays from the origin). The differential equation of the orthogonal family in the example is $y' = -1/(x+y)$ because in general two lines whose slopes are negative reciprocals of each other are perpendicular.

12. We talked about the Existence and Uniqueness Theorem and direction fields/diff eq's that did not qualify, but what does that mean? That there is no solution? What is the significance of this failure?

Uniqueness means there is at most one curve through an initial point (x_0, y_0). Existence means that there is (at least one) solution curve through an initial point (x_0, y_0). Together they say that there is exactly one solution to the equation with initial condition $y(x_0) = y_0$.

When the existence and uniqueness theorem does not apply, all bets are off. We have no way of making any conclusion. For instance, it could be that the conclusion is still true (exactly one curve through the bad point we have identified). It could also be true that there is no solution (no curve that satisfies the differential equation and passes through the point) — existence fails. Finally, it could be true that there are many curves through the point (uniqueness fails). For example, $y' = y/x$ has infinitely many solutions through the point $(0, 0)$ of the form $y = ax$ (all a), i.e. uniqueness fails. This is a bad point because at $x = 0$, the right side is undefined. On the other hand, at the points $(0, b)$ where $b \neq 0$, there is no solution at all (existence fails). You cannot solve $y' = y/x$ with initial condition $y(0) = b$ for $b \neq 0$.

Near the end of the course, we will be looking equations representing flow lines in which solution curves will meet at very interesting points (where something clearly goes wrong with the existence/uniqueness theory). These are called sources and sinks of the flows.

13. Lobster traps — what are $B(x)$ and $S(x)$? (super and subsolutions)

I used the metaphor of a lobster trap for any curve for which the direction field points consistently to one side of the curve. Because the solution must follow the direction field, it can only cross the curve in one direction not the other. It enters but never leaves. I will explain this rigorously below.

The straightforward name for these curves is that they are graphs of super or subsolutions. Any time you find a solution, it traps the rest of the solutions because they can't cross it. The difficulty with that is that is often nearly impossible to find an exact solution. Often, if you can find one solution you can find them all anyway. (An example where a specific solution ($y = 1$) helps is discussed below in Question 14.) On the other hand, there are many, many super and subsolutions. While these don't trap the solution from both sides, they do trap the solutions from one side. These can be used as funnels to demonstrate the asymptotics of solutions. Super and and subsolutions also help with error bounds and knowing when to trust numerics.

The example we gave was that $B(x) = x$ (B for big) was a supersolution to the equation $y' = y^2 - x^2$. Analytically, you check this by seeing that $B'(x) = 1$, whereas $B^2 - x^2 = x^2 - x^2 = 0$, so $B' < B^2 - x^2$. Graphically, this means that the direction field of the differential equation $y' = y^2 - x^2$ along the graph of the function B always points less steeply than B. Intuitively the solution curves can enter (along the horizontal tangent lines). We now give a proof that they can't leave. We reason by contradiction. Suppose for a solution curve $(x, y(x))$ is below $B(x)$, so that $y(x) < B(x)$ for $x_0 < x < x_1$ and then at x_1, $y(x_1) = B(x_1)$, that is, the two curves touch. We claim this is impossible (leads to a contradiction). Indeed, it follows that the slope of y at x_1 has to be **at least as steep** as the slope of B at x_1 (otherwise it can't catch up). This means $y'(x_1) \geq B/(x_1) > B(x_1)^2 - x_1^2 = y(x_1)^2 - x_1^2$. But this is a contradiction: $y'(x_1) = y(x_1)^2 - x_1^2$.

Similarly, we discussed $S(x) = -x$, a subsolution (S for small). S satisfies $S' = -1 < S^2 - x^2 = 0$, which means that when we graph it along with the direction field, the direction field has a steeper slope than S (S goes down, but the direction field is horizontal.) Similar reasoning shows that a solution curve can enter, not exit: It can enter the region $y > S(x)$ from below $S(x)$ and cross, but it cannot come from above the graph and cross so that it is underneath S. In summary, B traps solutions from above (sits on top of solutions) and S traps solutions from below (sits below solutions).

14. In the case $y' = y^2 - 2y + 1$, why does y' have to be positive? Why does $2y - 2$ have to be negative? Doesn't that require looking at the graph, which is what we are trying to avoid?

This is subtle. We have to be careful of what we know and how we know it. The key thing that makes this work is that $y = 1$ is a solution. We drew this on the graph, but we can also check it by easy arithmetic. This means that if we start a solution $Y(x)$ at $Y(x_0) < 1$, then it will never cross $y = 1$, i. e., $Y(x) < 1$ for all x. It follows that $2Y(x) - 2 < 0$ for all x. Furthermore, $Y'(x) = (Y(x) - 1)^2 > 0$. The reasoning I just presented does not depend on the picture, but it is much clearer what it all means if you draw a picture. Visualization helps in dealing with inequalities.

15. Convexity/concavity?

If $y'' > 0$, then y is concave up. If $y'' < 0$, then y is concave down. In the example above, $y' = y^2 - 2y + 1$, so that $y'' = 2yy' - 2y' = 2(y-1)y'$ and we discussed in detail in 13 above how we can see that $y - 1 < 0$ and $y' > 0$ so that $y'' < 0$ when y starts below $y = 1$. Similarly, the solution curves above $y = 1$ are all concave up ($y'' > 0$ in the case $y > 1$).

16. Questions about homework problems: is q constant in PS1 B1b? What is required in EP1.4 /1 and 9?

Yes, q was constant — too late to help. 2nd query answered directly by e-mail.