## 18.03 Spring 2009: Exam 3 Review

All but two of the following ten problem types are represented in some way on the third hour test. All ten may be on make-up tests and on the final exam. Just under half the test is about Fourier series and just over half is about the Laplace transform. The delta function appears in both parts.

### **Fourier Series** (just under half)

1. Find a Fourier series or Fourier sine or cosine series using the Fourier coefficient formulas, which will be included in the list of formulas on the exam.

2. Describe how a Fourier series extends a function outside of its initial range of definition, taking into account the period and/or odd or even symmetry. (No quarter periods.)

3. Operate on Fourier series: especially, differentiate them or use them to find periodic solutions to ordinary differential equations and to detect resonance/near resonance.

4. Take generalized derivatives with the help of graphs; integrate delta functions.

### **Laplace Transform** (just over half)

5. Derive a transform formula such as the ones for $\mathcal{L}(e^{at}f(t))$, $\mathcal{L}(f(at))$ or $\mathcal{L}(u(t-a)f(t-a))$ directly from the definition of the Laplace transform. (If present, this will be short.)

6. Use tables to take Laplace transforms, especially of initial value problems (including step function and delta function inputs).

7. Use tables to invert Laplace transforms, especially to solve initial value problems. (You can always multiply through to find the coefficients of a partial fraction decomposition, but it is faster to know the coverup method. You won't need the complex cover-up method on this exam.)

8. Calculate a convolution directly. Use weight functions and convolution to solve initial value problems (rest conditions only).

9. Use the Laplace transform to calculate a convolution indirectly.

10. Make pole diagrams: Plot the collection of points where poles are located in the complex $s$-plane. Describe the connection between poles of the Laplace transform $Y(s) = \mathcal{L}(y)$ and the long term behavior of $y(t) = \mathcal{L}^{-1}(y)$. In the case of simple poles, you will need to know the format of partial fractions, which leads, in turn, to the form of $y$, namely, $y(t)$ is a linear combination of exponentials/sines and cosines.[1]

The formulas that will be included in Exam 3 are on the next page.

[1] More generally, if $Y(s)$ has repeated roots in its denominator, then $Y(s)$ has double or multiple poles, and $y(t)$ has terms with powers of $t$ multiplying exponentials/sines and cosines. These terms are symptoms of resonance.

| Definition and properties of the Laplace transform | |
|---|---|
| $f(t)$ | $F(s) = \int_0^\infty f(t)e^{-st}\,dt$ |
| $af(t) + bg(t)$ | $aF(s) + bG(s)$ |
| $e^{at}f(t)$ | $F(s-a)$ |
| $tf(t)$ | $-F'(s)$ |
| For $a \geq 0$: | |
| $u(t-a)f(t-a)$ | $e^{-as}F(s)$ |
| Using $\mathcal{L}_+$ for a generalized derivative $f'$: | |
| $f'(t)$ | $sF(s) - f(0^+)$ |
| Using $\mathcal{L}_+$ for a generalized derivative $f''$: | |
| $f''(t)$ | $s^2F(s) - f(0^+)s - f'(0^+)$ |
| $f(t) * g(t)$ $= \int_0^t f(\tau)g(t-\tau)\,d\tau$ | $F(s)G(s)$ |

| Laplace transforms of some elementary functions | |
|---|---|
| $f(t)$ | $F(s)$ |
| $1$ | $\frac{1}{s}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ |
| $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ |
| For $a \geq 0$: | |
| $u(t-a)$ | $\frac{e^{-as}}{s}$ |
| For $a \geq 0$ (using $\mathcal{L}_-$ for $a = 0$): | |
| $\delta(t-a)$ | $e^{-as}$ |

**Coefficients of a Fourier series of period $2L$**

If $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$, then

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt.$$

In particular, if $f(t)$ is even,

$$b_n = 0 \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right) dt;$$

if $f(t)$ is odd,

$$a_n = 0 \quad \text{and} \quad b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right) dt.$$