18.03 Practice Problems on Fourier Series – Solutions

Graphs appear at the end.

1. **What is the Fourier series for** \(1 + \sin^2 t\)?

This function is periodic (of period \(2\pi\)), so it has a unique expression as a Fourier series. It’s easy to find using a trig identity. By the double angle formula, \(\cos(2t) = 1 - 2\sin^2 t\), so

\[
1 + \sin^2 t = \frac{3}{2} - \frac{1}{2}\cos(2t).
\]

The right hand side is a Fourier series; it happens to have only finitely many terms.

2. **Graph the function** \(f(t)\) **which is even, periodic of period** \(2\pi\), **and such that** \(f(t) = 2\) **for** \(0 < t < \frac{\pi}{2}\) **and** \(f(t) = 0\) **for** \(\frac{\pi}{2} < t < \pi\). **Is the function even, odd, or neither?**

Here is the graph of \(f(t)\). Note that there is only one way to extend the definition of \(f\) over all real \(t\) since \(f\) is specified to be even and periodic.

The function \(f(t)\) is even.

**Find its Fourier series in two ways:**

(a) **Use parity if possible to see that some coefficients are zero. Then use the integral expressions for the remaining Fourier coefficients.**

The function \(f(t)\) is even, so \(b_n = 0\) for all \(n > 0\). The only possibly nonzero coefficients are the \(a_n\)’s. Compute \(a_0\) first.

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 2 \, dt = 2.
\]

Now compute \(a_n\) for \(n > 0\). Since the function is even,

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \cos(nt) \, dt = \frac{2}{\pi} \int_{0}^{\pi/2} 2 \cos(nt) \, dt = \frac{4}{n\pi} \sin(nt) \bigg|_{0}^{\pi/2} = \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right)
\]

If \(n\) is even, this is always zero. If \(n\) is odd, then this alternates between \(\frac{4}{n\pi}\) when \(n\) of the form \(4k + 1\) and \(-\frac{4}{n\pi}\) when \(n\) is of the form \(4k + 3\).

The Fourier series is then

\[
f(t) = 1 + \frac{4}{\pi} \cos t - \frac{4}{3\pi} \cos(3t) + \frac{4}{5\pi} \cos(5t) - \frac{4}{7\pi} \cos(7t) + \cdots
\]

(b) **Express** \(f(t)\) **in terms of** \(sq(t)\), **substitute the Fourier series for** \(sq(t)\) **and use some trig identities.**

First we see that \(f\) can be expressed in terms of the standard square wave as

\[
f(t) = 1 + sq\left(t + \frac{\pi}{2}\right).
\]

Now (see overleaf) the Fourier series for \(sq(t)\) is

\[
 sq(t) = \frac{4}{\pi} \left( \sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots \right),
\]
so we can substitute this in to get the Fourier series for $f(t)$:

$$f(t) = 1 + \frac{4}{\pi} \left( \sin \left( t + \frac{\pi}{2} \right) + \frac{\sin \left( 3t + \frac{3\pi}{2} \right)}{3} + \frac{\sin \left( 5t + \frac{5\pi}{2} \right)}{5} + \cdots \right)$$

$$= 1 + \frac{4}{\pi} \cos t - \frac{4}{3\pi} \cos(3t) + \frac{4}{5\pi} \cos(5t) - \cdots.$$ 

Here we have used the angle addition formula for sine, and the same values of sine that we used in (a). This coincides with the answer we got for Part (a).

3. **Graph the function $h(t)$ which is odd and periodic of period $2\pi$ and such that $h(t) = t$ for $0 < t < \frac{\pi}{2}$ and $h(t) = \pi - t$ for $\frac{\pi}{2} < t < \pi$. What is its average value? Observe that $h'(t) = f(t) - 1$, where $f(t)$ is the function studied in Problem 2.**

The graph of $h(t)$ is a zigzag wave.

The function is odd, so its average is zero.

*Use these observations to find its Fourier series.*

We observe that the function $h(t)$ has derivative $f(t) - 1$, where $f(t)$ is the function described in Problem 1. The Fourier series for $f(t) - 1$ has zero constant term, so we can integrate it term by term to get the Fourier series for $h(t)$, up to a constant term given by the average of $h(t)$. Since $h(t)$ is odd, its average is 0. The rest of the series is computed below.

$$h(t) + c = \int (f(t) - 1)dt = \frac{4}{\pi} \int \left( \cos t - \frac{\cos(3t)}{3} + \frac{\cos(5t)}{5} - \cdots \right) dt$$

$$= \frac{4}{\pi} \left( \sin t - \frac{\sin(3t)}{9} + \frac{\sin(5t)}{25} - \cdots \right)$$

Since the right hand side has average value zero, $c = 0$.

4. **Explain why any function $F(x)$ is a sum of an even function and an odd function in just one way. Hint: $F_+(x) = \frac{F(x) + F(-x)}{2}$ is even. What is the even part of $e^x$? What is the odd part?**

First, $F_+(x)$ is even: $F_+(-x) = \frac{F(-x) + F(x)}{2} = F_+(x)$.

Now notice that $F(x) = F_+(x) + F_-(x)$ where $F_-(x) = \frac{F(x) - F(-x)}{2}$. I claim that $F_-(x)$ is odd: $F_-(x) = \frac{F(-x) - F(x)}{2} = -F_-(x)$.

So $F(x)$ is the sum of an even function and an odd function.

To show that this decomposition is unique, we suppose we have another decomposition $\tilde{F}_+(x) + \tilde{F}_-(x) = F(x)$, where $\tilde{F}_+(x)$ is even and $\tilde{F}_-(x)$ is odd.

Then $F_+(x) + F_-(x) = F(x) = \tilde{F}_+(x) + \tilde{F}_-(x)$, so $F_+(x) - \tilde{F}_+(x) = \tilde{F}_-(x) - F_-(x)$. But the left hand side is even and the right hand side is odd, so they both must be zero, which says that $F_+(x) = \tilde{F}_+(x)$ and $F_-(x) = \tilde{F}_-(x)$.

This decomposition might seem familiar from hyperbolic trig function formulas: The even part of $e^x$ is $\frac{e^x + e^{-x}}{2} = \cosh x$, and the odd part of $e^x$ is $\frac{e^x - e^{-x}}{2} = \sinh x$. 

Figure 1: Graph of $f(t)$ over three periods.

Figure 2: Graph of $h(t)$ over three periods.