18.02A Problem Set 6
Due Tuesday March 15 2011, 11:00 am in 2-108.

Lecture 15. Mar.8  
Line integrals in space, curl, exactness and potentials.
Lecture Preparation: Read notes V11, V12.1, post your questions and insights on the forum (by Tuesday 9am, tag #reading15), and try to solve the following problem (work due Tuesday 11am in class).

Problem A1. a) Calculate the work done by the force field \( \vec{F} = (xz + y)\hat{i} + zy\hat{j} + z\hat{k} \) on a particle moving along the path \( x = \cos t, \ y = \sin t, \ z = t \) from \((1, 0, 0)\) to \((1, 0, 2\pi)\).

b) Show that the force in (a) is not conservative. Hint: You can either apply a mixed derivative test or show that work depends on the path.

Further Homework: 6D/ 1, 2, 4, 5; 6E/ 1, 2, 3ab(ii) (both methods), 5.

Lecture 16. Mar.10  
Stokes’ theorem.
Lecture Preparation: Read Notes V13.1, V13.2 (or Simmons 21.5), post your questions and insights on the forum (by Thursday 9am), and try to solve the following problem(work due Thursday 11am in class).

Problem A2. a) Calculate the work \( \int_C \vec{F} \cdot d\vec{r} \) of the vector field \( \vec{F} = (xz + y)\hat{i} + zy\hat{j} + z\hat{k} \) counterclockwise along the unit circle in the xy-plane in two ways:

(i) directly  
(ii) by using Green’s theorem

b) Calculate curl \( \vec{F} \) and its flux \( \iint_S (\text{curl } \vec{F}) \cdot \hat{n} \, dS \) through the upper hemisphere \( S \) of radius one (that is \( S \) is given by \( x^2 + y^2 + z^2 = 1, \ z \geq 0 \)). By Stokes’ theorem, if you pick the normal pointing away from the origin, your result should be the same as in (a).

Further Homework: 6F/ 1, 2, 3, 5.

Lecture 17. Mar.11  
Stokes’ theorem continued; Applications.
Lecture Preparation: Read Simmons 21.6 (more details in notes V15.1-2)\(^1\), post your questions and insights on the forum (by Friday 9am), and try to solve the following problem (work due Friday 2pm in class).

Problem A3. Check that the electric field \( \vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \) of a charge \( Q \) at the origin indeed satisfies the integral form of Maxwell’s equations:

a) \( \oint_C \vec{E} \cdot d\vec{r} = 0 \) for every closed curve \( C \),

b1) \( \iint_S \vec{E} \cdot \hat{n} \, dS = 0 \) for every closed surface \( S \) that does not have the origin in its interior,

b2) \( \iint_S \vec{E} \cdot \hat{n} \, dS = \frac{Q}{\epsilon_0} \) for every sphere \( S \) of radius \( a \) around the origin.

Further Homework: 6G/1; 6H/1, 2.

Lecture 18, Mar.15 : Review  
Final: Mar.16, 7:30-9:30pm in 4-149

\(^1\)The Gauss-Coulomb law \( \text{div } \vec{E} = 4\pi \rho \) in the notes translates to the book by \( \rho = \frac{Q}{4\pi\epsilon_0} \). The notes also make Faraday’s law and Ampère’s law (without currents) more symmetric by using \( \vec{B}' = c\vec{B} \) as magnetic field, so that \( \text{curl } \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \) and \( \text{curl } \vec{B}' = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \).
Part A (16 points): Hand in problems A1–3 and the underlined problems listed above. (If you handed in A1-3 in class and did not pick it up again, it will automatically get graded.)

Part B (14 points)

Attempt to solve each part of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done. Write on the first page the names of all the people you consulted or with whom you collaborated and the resources you used.

Problem B1. (Tuesday, 2+1+2 points)
a) Compute (in terms of the constants $a$, $b$) the work done by the vector field

$$\vec{F} = (a \sin z + b xy^2) \hat{i} + x^2 y \hat{j} + (x \cos z - z^3) \hat{k}$$

along the portion of helix $x = \cos t$, $y = \sin t$, $z = t$ from $(1,0,0)$ to $(1,0,2\pi)$.

b) Compute curl $\vec{F}$. For which value(s) of $a$ and $b$ is the vector field $\vec{F}$ conservative?

c) Let $a$ and $b$ be the values you found above. Find a potential function for $\vec{F}$ using a systematic method, and verify the answer you found in part (a) using the fundamental theorem of calculus.

Problem B2. (Tuesday/Thursday, 2+2+3+2 points)

Consider the same tetrahedron as in Problem Set 5 B2, with vertices at $P_0 = (0,0,0)$, $P_1 = (1,0,1)$, $P_2 = (1,0, -1)$, and $P_3 = (1,1,0)$.

a) Say which orientation (order of vertices) of the boundary curve of each face is compatible with the choice of the normal vector pointing out of the tetrahedron.

b) Compute the work done by the vector field $\vec{F} = yz \hat{j} - y^2 \hat{k}$ around the boundary curve of the face $P_0P_1P_3$ directly using line integrals (using the orientation from part (a)).

c) Use Stokes’ theorem to compute the work done around each of the four faces (including the one you computed directly in part (b)). Use the orientations from part (a). (Note that you can avoid some calculations by re-using your answers from Problem Set 5.)

d) The sum of the four values you found in part (c) should be zero. Explain this in two different ways:

(i) geometrically, by considering the various line integrals that are being added together;
(ii) by using the divergence theorem to compute the flux of curl $\vec{F}$ out of the tetrahedron.