18.02A Problem Set 2
Due Tuesday Feb 15 2011, 11:00 am in 2-108.

Lecture 4. Feb.8 Gradient fields and potential functions.
Lecture Preparation: Read notes 21.2, V2.1, V2.2, try to solve the following problem A1 (due Tuesday 11am in class), and post on the forum (by Tuesday 9am with tag #reading4):

- What are you most confused by in the problem or reading?
- What interesting insight did you get from the reading / own thinking / other sources?
Alternatively: Help resolve someone else’s confusion by giving an explanation. (Do not explicitly solve the problem though.)

Problem A1: a) Try to find a function \( f \) such that \( \frac{\partial f}{\partial x} = 5y^2 \) and \( \frac{\partial f}{\partial y} = 3x \), by integrating both equations. Check your answer resp. explain why there cannot be such an \( f \). (Do not use the fundamental theorem or mixed derivative test here.)

b) Interpret the task of a) in terms of the vector field \( \vec{F} = 5y^2 \hat{i} + 3x \hat{j} \) (from Problem Set 1). Now use the fundamental theorem or mixed derivative test to explain your answer in a).

Further Homework: 4C/ 5a (by method 1 of V2), 5b (by method 2).

Lecture 5. Feb.10 Green’s theorem.
Lecture Preparation: Read 21.3, post your questions and insights on the forum (by Thursday 9am, tag #reading5), and try to solve the following problem (work due Thursday 11am in class).

Problem A2. Integrate the function \( f = \text{curl}(5y^2 \hat{i} + 3x \hat{j}) \) over the region between the parabola \( y = x^2 \) and the straight line \( y = 1 \) in two ways:

a) directly as double integral;
b) using Green’s theorem and the results from Problem Set 1, A2.

Further Homework: 4D/ 1abc, 2, 4, 5.

Lecture Preparation: Read Notes V3, V4.1-2, (and after class V4.3, V5), post your questions and insights on the forum (by Friday 9am, tag #reading6), and try to solve the following problem (work due Friday 2pm in class).

Problem A3. a) A (compressible) fluid is flowing over the plane with velocity field \( \vec{F} = (x\hat{i} + y\hat{j}) \frac{\text{kg}}{\text{m} \cdot \text{s}} \) (where \( x, y \) are measured in meter). Without using line integrals, calculate how much fluid in kilogram per second flows out of the rectangle with corners at \((0,0),(2,0),(0,1),(2,1)\).

b) The answer to a) can also be found by calculating the line integral \( \int_C x \, dy - y \, dx \) counter-clockwise along the rectangle. Check your answer by evaluating this integral, using Green’s theorem.

Further Homework: 4E/ 1ac, 2, 3, 5 (don’t use Green’s theorem); 4F/ 3, 4; 4G/ 1 (just consider a,c,e – we won’t talk about differentials)
Part A (16 points): Hand in problems A1–3 and the underlined problems listed above. (Now this gets graded for correctness, not just effort, so you may want to improve your earlier work. If your pre-class solution was already complete, just hand it in again. If you did not pick it up after class, it will automatically get to the grader.)

Part B (16 points)

Attempt to solve each part of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done. Write on the first page the names of all the people you consulted or with whom you collaborated and the resources you used. (Write 'none' if you did not use any collaboration or resources.)

Problem B1. (Tuesday, 1+2 points)

Continuing Problem B3 of Problem Set 1, we consider the vector field \( \vec{F}(x, y) = \frac{-y}{x^2 + y^2} \hat{i} + x \hat{j} \).

e) Show that the curl of \( \vec{F} \) is zero at any point of the plane where \( \vec{F} \) is defined (not just in the right half-plane \( x > 0 \)).

f) Since curl \( \vec{F} = 0 \), one would hope to find a potential for \( \vec{F} \) by integrating along suitable curves. For example, define the function \( f(x, y) \) by \( f(x, 0) := \int_1^x \vec{F} \cdot \hat{i} \, dx \) for \( x > 0 \) and \( f(x, y) := f(\sqrt{x^2 + y^2}, 0) + \int_C \vec{F} \cdot d\vec{r} \), where \( C \) is a circle segment between \((x, y)\) and the positive \( x \)-axis.

Evaluate these line integrals to obtain a formula for \( f \). Explain the complication with this formula and why this does not make \( \vec{F} \) conservative on the whole plane.

Finally, consider the region obtained from the plane by cutting out the graph of \( y = x^2 \), \( x \leq 0 \). Can you find a potential function for \( \vec{F} \) on this region?

Problem B2. (Tuesday, 1+2 points)

a) Calculate the curl of \( \vec{F} = r^n \hat{r} = r^n(x \hat{i} + y \hat{j}) \) for all integers \( n \). (You can simplify the differentiation by checking that \( r_x = x/r \) and \( r_y = y/r \).)

b) For each \( n \) for which curl \( \vec{F} = 0 \), find a potential \( g \) such that \( \vec{F} = \nabla g \). (Hint: look for a potential of the form \( g = g(r) \), with \( r = \sqrt{x^2 + y^2} \). Watch out for a certain negative value of \( n \) for which the formula is different.)

Problem B3. (Thursday, 5 points)

Let \( C \) be a curve that starts somewhere on the positive \( x \)-axis, ends somewhere on the line \( y = 1 \), and is contained in the half strip \( 0 \leq y \leq 1, x \geq 0 \). Show that the work of the vector field \( \vec{F} = (2xy^2 - 2xy) \hat{i} + (2x^2y - x^2 + 3y^2) \hat{j} \) along \( C \) is always 1. (Hint: Apply Green’s theorem to the region between \( C \) and the \( y \)-axis.)

Problem B4. (Friday, 1+2+2 points)

We consider the vector field \( \vec{F} = r^{-2} \hat{r} = r^{-2}(x \hat{i} + y \hat{j}) \).

a) Calculate the divergence of \( \vec{F} \).
(Divergence is defined as \( \text{div}(M \hat{i} + N \hat{j}) = M_x + N_y \), see the notes V4.2.)

b) Find the flux of \( \vec{F} \) outwards through the circle of radius \( a \) centered at the origin.

c) Find the flux of \( \vec{F} \) outwards through any circle centered at \((1, 0)\) of radius \( R \neq 1 \). Consider the cases \( R > 1 \) and \( R < 1 \) separately. Explain your answers with diagrams. (Hint: Instead of setting up another line integral, combine the normal form of Green’s theorem cleverly with a) and b.)