Problem 1. *(Short answer; 5 pts each)* You are not required to show detailed work for these questions unless specifically asked to do so.

a) Suppose that \( \mathbf{F} = \nabla f(x, y, z) \), and \( f(P) = 0 \) for some point \( P \). Describe all of the points \( Q \) that satisfy \( \int_{P}^{Q} \mathbf{F} \cdot d\mathbf{r} = 2 \).

b) If \( f(x, y) \) is a scalar function and \( \mathbf{F} = M_1(x, y) \mathbf{i} + M_2(x, y) \mathbf{j} \) is a vector field, prove the identity

\[
\text{div} (f \mathbf{F}) = f(\text{div} \mathbf{F}) + (\nabla f) \cdot \mathbf{F}.
\]

c) Let \( \mathbf{F} = \nabla(\cos(x + 3yz)) \). What is the maximum possible value of \( \int_{c} \mathbf{F} \cdot d\mathbf{r} \), where \( c \) is any path? *This only depends on certain properties of the cosine function...*

d) A surface \( S \) is given by the parameterization

\[
x = u, \quad y = v, \quad z = f(u, v).
\]

What is the area differential \( dA \) (i.e., what expression fills in the blank: \( A = \left( \int_{S} \ldots \right) dA \)?)
e) Write down the statement of Green's Theorem (be as precise as you can). 

f) If \( \mathbf{F} = 2x \mathbf{i} + 2y \mathbf{j} \), find the work done by \( \mathbf{F} \) around the counterclockwise unit circle (argue geometrically to avoid calculating a line integral).

Assume \( \mathbf{F} \) has singularities at \( P_1 \) and \( P_2 \) but elsewhere satisfies and \( \mathbf{F} = 0 \).

g) Suppose that \( \int_{c_1} \mathbf{F} \cdot d\mathbf{r} = 3 \) and \( \int_{c_2} \mathbf{F} \cdot d\mathbf{r} = -2 \), where \( c_1 \) and \( c_2 \) are very small positively oriented paths around points \( P_1 \) and \( P_2 \). Calculate \( \int_{c} \mathbf{F} \cdot d\mathbf{r} \) for the path given in the figure.

h) Let \( R \) be a region in space with density \( \delta(x, y, z) \). Write down an integral expression that gives the \( x \)-coordinate of \( R \)'s center of mass.
Problem 2. (20 pts: 15+5) Let $R$ be the region between the surfaces $z = 2\sqrt{x^2 + y^2}$ and $z = 3 - \sqrt{x^2 + y^2}$.

a) Use cylindrical coordinates to calculate the volume of $R$.

b) Write down (but do not evaluate) an integral expression for the moment of inertia of $R$ around the $z$-axis. $\left(\sigma = 1\right)$
Problem 3. (30 pts: 10+10+10) Let

\[ F = \left( ax \sin y - \frac{x^2}{2} \right) \hat{i} + (x^2 + 1) \cos y \hat{j} \]

a) For what value(s) of \( a \) is \( F \) conservative?

b) Using your answer to part a), find a potential function by any method.

c) For what value(s) of \( a \) is \( \text{div} \, F = 0? \)
Problem 4. (20 pts: 10+10) Let $c$ be the counterclockwise path that follows the $x$-axis from $(-1,0)$ to $(1,0)$, and then a circular arc back to $(-1,0)$.

a) Calculate $\int_c x \, dy - 2y \, dx$ directly (you may need the value of the integral $\int_0^\pi \sin^2 \theta \, d\theta = \frac{\pi}{2}$).

b) Calculate $\int_c x \, dy - 2y \, dx$ by using the closed form of Green's Theorem.

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Problem 5. (20 points) Let $Q$ be the solid cone with boundaries $z = \sqrt{x^2 + y^2}$ and $z = 1$. In parts a) through c), set up but do not evaluate the triple integral $\iiint_Q 1 \, dV$ for the volume of the cone in the given coordinate system.

(a) Cartesian $xyz$-coordinates.

(b) Cylindrical coordinates.

(c) Spherical coordinates.

(d) Now choose one of the three coordinate systems above, and finish evaluating the triple integral to calculate the volume of the cone.
Problem 6. (15) Let $R$ be the portion of the disk of radius 2 lying in the first quadrant, between the $y$-axis and the line $y = x$. Express the integral $\int \int_R \frac{x}{y} \, dA$ in terms of the variables $u = x/y$ and $v = x^2 + y^2$, and evaluate it.