Math 18.02 (Spring 2009): Lecture 5
Planes. Parametric equations of curves and lines

February 12

Reading Material: From *Simmons*: 17.1 and 17.2.

Last time: Square Systems. Word problem. How many solutions? Equations of planes
Today: Planes. Parametric equations of curves and lines

2 Equations of Planes

In this section we recall the *analytic* description of a plane through a point $P_0 = (x_0, y_0, z_0)$ and perpendicular to a vector $\vec{N} = (a, b, c)$. This plane is described by all points $P = (x, y, z)$ such that the vector $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ is perpendicular to the given vector $\vec{N}$:

$$\vec{N} \perp \overrightarrow{P_0P} \iff \vec{N} \cdot \overrightarrow{P_0P} = 0.$$

Since

$$\vec{N} \cdot \overrightarrow{P_0P} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

it follows that the (*analytic*) equation for our plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{2.1}$$

Exercise 1. *Find a plane containing* $P_1 = (1, 2, 3), P_2 = (1, 4, 4)$ *and* $P_3 = (0, 2, 6)$.

Method: Find a vector $\vec{N}$ normal to two vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ by computing $\vec{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$. Use $\vec{N}$ and any of P_1, P_2, P_3 to write the appropriate equation like in (2.1). Now the details:

To write equation (2.1) we take the vector $\vec{N}$ we just computed and the point $P_2 = (1, 4, 4)$ for example[1] and we get

that can also be written as $6x - y + 2z = 10$. Let's now check if P_1, P_2 and P_3 belong to this plane:

Exercise 2. *Given the two planes*

$$3x + 4y = 10 \qquad \leftarrow \textit{note a plane in 3D parallel to the z-axis}$$
$$2x - y + 2z = 5$$

find the angle between them.

[1] Here one can use either one of the other two given points, the equation for the plane at the end will be the same!

Method: Consider the general situation of two planes

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$

with angle θ between them:

Observe that θ is the same as the angle between the two normals $\overrightarrow{N}_1$ and $\overrightarrow{N}_2$ to each plane respectively. We know that the coefficients in the equation of a plane represent the coordinates of a normal vector to the plane itself. Hence

Now recall that from the definition of dot product one can deduce that

and the problem is solved.

Let's go back to Exercise 2. From the two planes

$$3x + 4y = 10$$
$$2x - y + 2z = 5$$

we obtain the corresponding normals

moreover

3 Parametric equations of curves and lines

In Math 18.01 you described curves as

1. either the graph of a function $y = f(x)$ [explicit]

2. or as the points on the plane such that $F(x, y) = 0$ [implicit]

These two ways were okay in 2D but *not* in 3D! In fact the graph of a two variable function $z = f(x, y)$ gives a *surface* not a curve:

In 3D in order to describe a curve we need the so called *Parametric Equations*

Definition. *A* **curve** γ *in 3D is defined by the points* (x, y, z) *such that*

$$\begin{aligned} x &= f(t) \\ y &= g(t) \\ z &= h(t) \end{aligned}$$

with the parameter t *ranging in a certain interval* $[a, b]$*. The functions* $f(t), g(t)$ *and* $h(t)$ *are called parametric equations.*

2D example: Consider the curve γ described by the parametric equations

$$\begin{aligned} &x = \cos t \\ &y = \sin t, \quad \text{for } t \text{ in } [0, 4\pi]. \end{aligned}$$

3D example: Consider the curve γ described by the parametric equations

$$\begin{aligned} &x = \cos t \\ &y = \cos t \\ &z = t, \quad \text{for } t \text{ in } [0, 4\pi]. \end{aligned}$$

Equations for lines in 3D

Typical problem: Given a point P_0 and a vector $\vec{V}$, find the line through P_0 and parallel to $\vec{V}$.

In coordinates: $P_0 = (x_0, y_0, z_0)$ and $\vec{V} = (a, b, c)$ are given.

A generic point $P = (x, y, z)$ is on the line if

$$\overrightarrow{P_0P} \text{ is parallel to } \vec{V}$$

or in other words

$$\overrightarrow{P_0P} = t\vec{V}$$

for any scalar t. In coordinates this is written as

$$\begin{aligned} x - x_0 &= ta \\ y - y_0 &= tb \\ z - z_0 &= tc \end{aligned}$$

or equivalently

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}$$

and these are called the **parametric equations** of a line through $P_0 = (x_0, y_0, z_0)$ and parallel to $\vec{V} = (a, b, c)$.

Exercise 3. *Find the line through the two points* $P_0 = (1, 2, 3)$ *and* $P_1 = (0, 1, 0)$.

Method: Let's consider the general case $P_0 = (a_1, a_2, a_3)$ and $P_1 = (b_1, b_2, b_3)$.

Observe that the vector

$$\overrightarrow{P_0P_1} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

is by construction parallel (actually on top of it!) to the line we are looking for. Then one can use the parametric equations above and obtain

$$\begin{aligned} x &= a_1 + t(a_1 - b_1) \\ y &= a_2 + t(a_2 - b_2) \\ z &= a_3 + t(a_3 - b_3). \end{aligned}$$

Going back to Exercise 3 we then have the parametric equations

$$
\begin{aligned}
x &= 1 - t \\
y &= 2 - t \\
z &= 3 - 3t.
\end{aligned}
$$

Handy Fact: If you want to describe only the segment of line from P_0 to P_1 then you restrict the parameter in the interval $[0, 1]$.

Study Guide 1. *Answer the following questions:*

- *You have now two ways to describe a line in 3D: as the set of solutions to a system of two linear equations (intersection of two planes) or as a curve given by three very special parametric equations as described above. Can you see a connection between the two?*

- *When a curve is given certainly the parametric equations are essential. But it is also very important to describe the range of the parameter. Can you give an example of this fact?*