

The Gowers Norm in the Testing of Boolean Functions

by

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Abstract

A property tester is a fast, randomized algorithm that reads only a few entries of the input, and based on the values of these entries, it distinguishes whether the input has a certain property or is “different looking” from any input having this property. Furthermore, we say that a property tester has completeness c and soundness s if it accepts all inputs having the property with probability at least c and accepts “different” inputs with probability at most $s + o(1)$.

In this thesis we present two property testers for boolean functions on the boolean cube $\{0, 1\}^n$. We summarize our contribution as follows.

- We present a new dictatorship test that determines whether the function is a dictator (of the form $f(x) = x_i$ for some coordinate i), or a function that is an “anti-dictator.” Our test is “adaptive,” makes q queries, has completeness 1, and soundness $O(q^3) \cdot 2^{-q}$. Previously, a dictatorship test that has soundness $(q + 1) \cdot 2^{-q}$ is achieved by Samorodnitsky and Trevisan, but their test has completeness strictly less than 1. Furthermore, the previously best known dictatorship test from the PCP literature with completeness 1 has soundness $2^{O(\sqrt{q})-q}$. Our contribution lies in achieving both perfect completeness and low soundness simultaneously.
- We consider properties of functions that are invariant under linear transformations of the boolean cube. Previous works, such as linearity testing and low-degree testing, have focused on linear properties. The one exception is a test due to Green for “triangle freeness”: a function f satisfies this property if $f(x), f(y), f(x + y)$ do not all equal 1, for any pair $x, y \in \{0, 1\}^n$. We extend this test to a more systematic study and consider non-linear properties that are described by a single forbidden pattern. Specifically, let M denote an r by k matrix over $\{0, 1\}$. We say that a function f is M -free if there are no $\vec{x} = (x_1, \dots, x_k)$, where $x_1, \dots, x_k \in \{0, 1\}^n$ such that $f(x_1), \dots, f(x_k) = 1$ and $M\vec{x} = 0$. We analyze a test that determines whether a function is M -free or far from being one. Our test makes k queries, has completeness 1, and soundness bounded away from 1. The technique from our work leads to alternate proofs

that some previously studied linear properties are testable, albeit with worse parameter.

Independently, Král', Serra, and Vena have also established that M -freeness is testable when M is represented by a graph, and subsequently, these three authors and Shapira both independently showed that M -freeness is testable for any matrix M .

Our results, though quite different in terms of context, are connected by a similar set of techniques used in analyzing the algorithms. The analysis relies on the machinery of the Gowers uniformity norm, a recent and powerful tool in additive combinatorics.

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