The Clique Number of the Graph of Pairwise Sums and Products is 3 or 4

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May 17, 2004

Abstract

Let $G_+^\times$ be the graph with vertex set $\mathbb{N}$, with n and m adjacent if $n \neq m$ and if there exists positive integers x and y such that $x+y=n$ and $xy=m$ or $x+y=m$ and $xy=n$. The question of whether the chromatic number $\chi(G_+^\times)$ is finite is considered one of the few outstanding problems on partition regularity. We prove that the largest complete subgraph of $G_+^\times$ is either a K_3 or K_4. We prove that there is no K_4 in $G_+^\times$ with 2 vertices that are consecutive integers, though there are infinitely many K_4 with an edge deleted that have 2 pairs of vertices that are consecutive integers. We also give a 3-coloring of the vertices of $G_+^\times$ such that no triangle in $G_+^\times$ has all its vertices of the same color. These new results support our conjecture that $\chi(G_+^\times)$ is finite.

1 Introduction

Basic, relevant graph theory terms are defined in Appendix 1.

In 1916, while trying to prove Fermat's Last Theorem, Issai Schur proved what is arguably the first result in Ramsey theory [13]. Schur's theorem states the following: for any given t, there exists a least $S(t)$ for which every t-coloring of the positive integers from 1 to $S(t)$ must contain x, y, z of one color with $x+y=z$. It follows from Schur's theorem that for any given t, there exists a least $P(t)$ for which every t-coloring of the positive integers from 1 to $P(t)$ must contain x, y, z of one color with $xy=z$ [10]. This is because $x+y=z$ if and only if $2^x 2^y = 2^z$. This gives the bound $P(t) \leq 2^{S(t)}$.

It follows that for any finite coloring of the positive integers, there exists x and y such that x, y, and $x+y$ are all the same color. Likewise, for any finite coloring of the positive integers, there exists x and y such that x, y, and xy are all the same color. Neil Hindman has asked, in several different articles, whether a stronger result that combines Schur's theorem for sums and products holds [5], [7], [8], [9], [10].

Problem 1 *For every finite coloring of the positive integers, does there exist x and y such that x, y, $x+y$, and xy are all the same color?*

This problem has been open for at least 25 years, with few results supporting conclusions in either direction. It appears that every time the problem has been stated, the authors have believed that for every finite coloring of the positive integers there exists x and y such that x, y, $x+y$, and xy are all the same color. Ron Graham [7] proved that for every 2-coloring of the integers [1, 252], there exist x and y such that x, y, $x+y$, and xy are all the same color, and that 252 is the smallest positive integer for which this is true.

Besides Hindman's problem, there are many other open problems in number theory which mix addition and multiplication, including the famous Goldbach's conjecture. To date, no decent method for solving such problems has been developed.

In the recent survey "Open Problems in Partition Regularity," Hindman et al. [10] state that "it is rather extraordinary" that the following weaker version of Problem 1 has not even been resolved.

Problem 2 *For every finite coloring of the positive integers, does there exist x and y such that not both x and y are 2, and $x+y$ and xy are both the same color?*

We do not allow both x and y to be 2 because $2+2=4=2^2$, and allowing $x=y=2$ would trivialize the problem. In investigating this problem, it is natural to define the following graph: let $G_+^\times$ be the graph with vertex set $\mathbb{N}$, with n and m adjacent if $n \neq m$ and if there exists positive integers x and y such that $x+y=n$ and $xy=m$ or $x+y=m$ and $xy=n$.

The chromatic number $\chi(G_+^\times)=r$ is the smallest positive integer such that there is an r-coloring of the positive integers with x and y not both 2, and xy and $x+y$ not both the same color. Therefore, Problem 2 is equivalent to determining whether or not $\chi(G_+^\times)=\infty$. Halbeisen [6] recently showed that $\chi(G_+^\times)\geq 4$ by exhibiting a subgraph of $G_+^\times$ with chromatic number 4.

In Section 2, we prove that if $x<y<z$ are vertices of a triangle in G, then

$$\frac{x(x+6)^{\frac{1}{2}}}{2} \leq z \leq \frac{x^2}{4} \tag{1}$$

Using this result, we prove in Section 3 that $G_+^\times$ does not have a K_5 subgraph. Because we know that $G_+^\times$ contains triangles, such as the one on vertices 6, 7, 8, it follows that the largest complete subgraph of $G_+^\times$ is either K_3 or K_4. Using stronger results concerning triangles of $G_+^\times$ with two vertices that are consecutive positive integers, we prove that $G_+^\times$ does not have a K_4 subgraph with 2 of its vertices being consecutive integers, though there are infinitely many K_4 subgraphs with an edge deleted with 2 pairs of vertices of the K_4 subgraph are consecutive integers. Finally, we give a 3-coloring of $\mathbb{N}$ such that no three vertices of a triangle in $G_+^\times$ are all the same color.

2 Inequalities on Triangles in $G_{+}^{\times}$

In this section, we prove several lemmas which together imply Equation 1. The first result gives bounds on adjacent vertices of $G_{+}^{\times}$.

Lemma 1 *Assume a, $b \in \mathbb{N}$ and $x = a + b$, $y = ab$ are adjacent vertices of $G_{+}^{\times}$, with $x \neq y$. Then,*

$$x - 1 \leq y \leq \frac{x^2}{4}$$

If neither a nor b is one, then $y > x$.

Proof: Since we assume that $x \neq y$, we don't allow the special case $a = b = 2$.

We have $\frac{x^2}{4} - y = \frac{(a+b)^2}{4} - ab = \frac{(a-b)^2}{4} \geq 0$, so $y \leq \frac{x^2}{4}$.

We may assume without loss of generality that $a \leq b$, so $x = a + b \geq 2a$. Since $b = x - a$, then $y = a(x - a) = ax - a^2 \geq ax - \frac{ax}{2} = \frac{ax}{2}$. If $a > 2$, then $y > x$. If $a = 2$, then $b > a = 2$ and $x = a + b \geq 5$. This implies that $y = 2x - 4 \geq x + 1$. If $a = 1$, then $y = x - 1$, so we are done. □

The following lemma will be very useful in the proofs of several later lemmas.

Lemma 2 *Let x be an integer and a and b be distinct positive integers with $a + b < x$. We have $a < b$ if and only if*

$$a(x - a) < b(x - b)$$

Proof: Assume $a < b$. Since $a + b < x$, then $(b - a)(a + b) < x(b - a)$. Rearranging, we have $a(x - a) < b(x - b)$.

Assume $a(x-a) < b(x-b)$ and for contradiction, that $a > b$. We can rewrite $a(x - a) < b(x - b)$ as $(b - a)(b + a) < x(b - a)$, so $b + a > x$ which contradicts the assumption that $a + b < x$. □

We now show that there are no triangles in $G_{+}^{\times}$ with vertices x, y, z, no two of which are consecutive integers, with x, y, z relatively "close" together.

Theorem 1 *If x, y, z are vertices of a triangle in $G_{+}^{\times}$ with $x+1 < y$, $y+1 < z$, then*

$$\frac{x(x+6)^{\frac{1}{2}}}{2} \leq z \leq \frac{x^2}{4}$$

Proof: The fact that $z \leq \frac{x^2}{4}$ follows immediately from Lemma 1.

Since no two of x, y, and z are consecutive, it follows that the larger number in every edge is the product and the smaller is the sum. Without loss of generality, we may write
$x = b + (x - b)$ and $z = b(x - b)$ with $1 < b \leq x - b < x$,

$x = a + (x - a)$ and $y = a(x - a)$ with $1 < a \leq x - a < x$, and
$y = c + (y - c)$ and $z = c(y - c)$ with $1 < c \leq y - c < y$.
Combining these equations,

$$b(x - b) = z = c(a(x - a) - c).$$

Simplifying,

$$x(ca - b) = ca^2 + c^2 - b^2.$$

Since $a(x-a) = y < z = b(x-b)$, we know that $a \neq b$. Moreover, since $a \leq x-a$ and $b \leq x - b$, we have $a + b < x$. By Lemma 2 we conclude that $a < b$. By Lemma 1, since $a \neq 1$, $y = a(x-a) > x$, we have $b(y-b) > b(x-b) = z = c(y-c)$. Also, since $b \leq x - b$, then $b \leq \frac{x}{2} < \frac{y}{2}$. Since $c \leq y - c$, we have $c \leq \frac{y}{2}$. Thus $b + c < y$. Therefore, by Lemma 1, we know that $c < b$.

Case 1: $b = ca$. In this case, we have from (2) that $0 = ca^2 + c^2 - b^2 = ca^2 + c^2 - (ca)^2$. Since $c > 0$, we may divide out by c and solve for c:

$$c = 1 + \frac{1}{a^2 - 1}$$

Since c is a positive integer, $a^2 - 1$ must be a factor of 1, so $a^2 = 0$ or 2, which contradicts a being a positive integer.

Case 2: $b \neq ca$. We solve for x in (2):

$$x = \frac{ca^2 + c^2 - b^2}{ca - b} = a + \frac{ba^2 + c^2 - b^2}{ca - b}$$

Case 2a: $b > ca$. In this case (since a, b, $c \geq 2$ and $b - ca \geq 1$)

$$x = \frac{ca^2 + c^2 - b^2}{ca - b} = a + \frac{-ba - c^2 + b^2}{b - ca}$$

$$\leq a + b^2 - ba - c^2 = b^2 + a(1 - b) - c^2$$

$$\leq b^2 - 2b - 2 \leq b^2 - 6.$$

Case 2b: $b < ca$. In this case (since a, $c \leq b - 1$, $b \geq 3$ and $ca - b \geq 1$)

$$x = \frac{ca^2 + c^2 - b^2}{ca - b} = a + \frac{ba + c^2 - b^2}{ca - b}$$

$$\leq a - b^2 + ba + c^2 \leq (b - 1) - b^2 + b(b - 1) + (b - 1)^2$$

$$= b^2 - 2b \leq b^2 - 6.$$

In both Case 2a and 2b, $(x + 6)^{\frac{1}{2}} \leq b$ and

$$z = b(x - b) \geq b(x - \frac{x}{2}) = \frac{bx}{2} \geq \frac{x(x + 6)^{\frac{1}{2}}}{2}.$$

$\square$

Lemma 3 *If $x < x+1 < y$ are vertices of a triangle in $G_+^\times$, then*

$$\frac{x^2+2x}{6} \leq y \leq \frac{x^2}{4}$$

Proof: Case 1: $y = x+2$. In this case, there must exist a with $1 < a < x-a < x$ such that $a(x-a) = x+2$. But since $a \geq 2$, $2(x-2) \leq a(x-a)$, so $2(x-2) \leq x+2$. Therefore, $x \leq 6$. The only such triangle is 6, 7, 8. In that case, $x = 6$ and $y = 8$, and we see that $\frac{6^2+12}{6} = 8 \leq y = 8 < 9 = \frac{6^2}{4}$.

Case 2: $y > x+2$. In this case, there must exist a and b with $a(x-a) = y = b(x+1-b)$ and $1 < a \leq x-a < x$, $1 < b \leq x-b < x$. From Lemma 1, we have $y \leq \frac{x^2}{4}$. Since $a(x+1-a) > a(x-a) = b(x+1-b)$, and $b \leq \frac{x+1}{2}$, $a \leq \frac{x}{2}$, then $a+b < x+1$. Applying Lemma 2, we have that $a > b$. Solving the equation $a(x-a) = b(x+1-b)$ for x, we have $x = \frac{b}{a-b} + a + b \leq b + a + b \leq 3a - 2$. Therefore, $a \geq \frac{x+2}{3}$. Since $a \leq \frac{x}{2}$, then

$$y = a(x-a) \geq a(x - \frac{x}{2}) = \frac{ax}{2} \geq \frac{x^2+2x}{6}$$

□

Lemma 4 *The positive integers $x < y-1 < y$ are vertices of a triangle in $G_+^\times$ if and only if $x = 6$ and $y = 8$, or $x \geq 6$ is even and $y = \frac{x^2}{4}$.*

Proof: As in Case 1 of Lemma 3, if $x = y-2$ then the only triangle with all three integers consecutive has vertices 6, 7, 8, so $x = 6$ and $y = 8$ in this case.

If $x < y-2$, then there exists positive integers a and b such that $y-1 = a(x-a)$, $y = b(x-b)$, $1 < a \leq x-a < x$, and $1 < b \leq x-b < x$. We also have $a \neq b$, $a \leq \frac{x}{2}$, $b \leq \frac{x}{2}$. Since $b(x-b) = a(x-a)+1 > a(x-a)$ and $a+b < x$, then by Lemma 2 we have that $b > a$. We can rearrange the equation $b(x-b) = a(x-a)+1$ as

$$(b-a)(x-(b+a)) = 1$$

Since $b-a > 0$ and $b-a$ is a factor of 1, then $b-a = 1$. So $x-(b+a) = 1$, and substituting for a, we get $b = \frac{x}{2}$. Therefore, $y = b(x-b) = \frac{x^2}{4}$. If x were odd, then y would not be an integer. Therefore, x must be even.

If $x \geq 6$ is even and $y = \frac{x^2}{4}$, then x, $y-1$, and y are the vertices of triangle:
$x = \frac{x}{2} + \frac{x}{2} = (\frac{x}{2}-1) + (\frac{x}{2}+1)$,
$y = (\frac{x}{2})(\frac{x}{2})$, and
$y-1 = (\frac{x}{2}-1)(\frac{x}{2}+1)$.
The vertices y and $y-1$ are trivially adjacent. □

3 Main Result

Theorem 2 *There are no K_4 subgraphs of $G_+^\times$ that contain two consecutive integers as vertices.*

Proof: Assume x, $x+1$, y, z are vertices of K_4 in $G_+^\times$. Without loss of generality, $z > y$.

Case 1: $y < x$. In this case, by Lemma 4, either $y = 6$ and $x+1 = 8$, or $y \geq 6$ and $x + 1 = \frac{y^2}{4}$.

Case 1a: If $y = 6$ and $x + 1 = 8$, then by Lemma 1, $5 < 4\sqrt{2} \leq z \leq \frac{6^2}{4} = 9$. So $z = 7$, but 7 and 9 aren't adjacent.

Case 1b: $y \geq 6$ and $x+1 = \frac{y^2}{4}$, so if $z > x+1$, then y and z can't be adjacent by Lemma 1. So $z < x$, and applying Lemma 4, $z = y$, which is a contradiction.

Case 2: $z > y > x + 1 > x$. By Lemma 3, $\frac{x^2+2x}{6} \leq y \leq \frac{x^2}{4}$ and $\frac{x^2+2x}{6} \leq z \leq \frac{x^2}{4}$. If $z = y + 1$, then applying Lemma 4, x and $x + 1$ would have to be equal. Thus $z > y+1$ and there exists a positive integer $a > 1$ for which $z = a(y-a)$. By Lemma 2,

$$\frac{x^2}{4} \geq z = a(y-a) \geq 2(y-2) = 2y - 4 \geq 2(\frac{x^2+2x}{6}) - 4$$

The inequality $\frac{x^2}{4} \geq 2(\frac{x^2+2x}{6}) - 4$ fails for $x > 4$. Since the vertices 1, 2, 3, 4 are not vertices of a complete graph on 4 vertices, then there are no cliques with 4 vertices that contain two consecutive integers. □

If $x_1 < x_2 < x_3 < x_4 < x_5$ are vertices of a K_5 in $G_+^\times$, then $x_{i+1} > x_i + 1$ by Theorem 2. We now can apply Theorem 1 to show that no K_5 are in $G_+^\times$.

Theorem 3 *The graph $G_+^\times$ doesn't have a K_5 subgraph.*

Proof: Assume $x_1 < x_2 < x_3 < x_4 < x_5$ are vertices of a K_5 in $G_+^\times$. Applying Theorem 1 to the triangle with vertices x_1, x_2, x_3: $x_3 \geq \frac{x_1(x_1+6)^{\frac{1}{2}}}{2}$. Since x_5 is adjacent to x_1, then $x_5 \leq \frac{x_1^2}{4}$. Applying Theorem 1 to the triangle x_3, x_4, x_5:

$$\frac{x_1^2}{4} \geq x_5 \geq \frac{x_3(x_3+6)^{\frac{1}{2}}}{2} \geq \frac{\frac{x_1(x_1+6)^{\frac{1}{2}}}{2}(\frac{x_1(x_1+6)^{\frac{1}{2}}}{2}+6)^{\frac{1}{2}}}{2}$$

The inequality

$$\frac{x_1^2}{4} \geq \frac{\frac{x_1(x_1+6)^{\frac{1}{2}}}{2}(\frac{x_1(x_1+6)^{\frac{1}{2}}}{2}+6)^{\frac{1}{2}}}{2}$$

implies that

$$x_1 \geq (x_1+6)^{\frac{1}{2}}(\frac{x_1(x_1+6)^{\frac{1}{2}}}{2}+6)^{\frac{1}{2}}$$

Squaring both sides, we arrive at the equation

$$x_1^2 \geq (x_1+6)(\frac{x_1(x_1+6)^{\frac{1}{2}}}{2}+6).$$

But $x_1+6 > x_1$ and $x_1 \geq 1$ so $\frac{x_1(x_1+6)^{\frac{1}{2}}}{2} \geq \frac{x_1\sqrt{7}}{2} > x_1$. Thus $(x_1+6)(\frac{x_1(x_1+6)^{\frac{1}{2}}}{2}+6) > x_1^2$, a contradiction. □

Theorem 4 *There is a 3-coloring of the vertices of $G_+^\times$ such that no triangle of $G_+^\times$ has all 3 vertices the same color.*

Proof: Let $x_1 = 7$ and for $i \geq 1$, let $x_{i+1} = \lfloor \frac{x_i(x_i+6)^{\frac{1}{2}}}{2} \rfloor$. We first partition the integers into intervals:

Let $I_1 = [1, x_1)$, and for $i > 1$, let $I_i = [x_{i-1}, x_i)$.

If $i \equiv 0 \pmod 3$, let every positive integer in I_i be colored green.

If $i \equiv 1 \pmod 3$, let every positive integer in I_i be colored red.

If $i \equiv 2 \pmod 3$, let every positive integer in I_i be colored blue.

We now show that no 3 vertices $x < y < z$ in a triangle have the same color. It is clear from a quick check that none of the integers in I_1 are in a K_3 with red vertices. If $x < y < z$ are all the same color, and x, y, $z \in I_i$, then $z < \frac{x(x+6)^{\frac{1}{2}}}{2}$, and so from Lemma 1, Lemma 3, and Lemma 4, no such triangle exists. If $x \in I_i$ and $z \notin I_i$, then $z \in I_{i+3n}$ for some natural number n. So $x_i > x$, and $z \geq x_{i+2}$. By definition, $x_{i+2} > x_{i+1}(x_{i+1}+6)^{\frac{1}{2}} - 1$, and $x_{i+1} > x_i(x_i+6)^{\frac{1}{2}} - 1$. From these two inequalities and $x_i > 7$ for $i > 1$, it follows that $z \geq x_{i+2} \geq \frac{x_i^2}{4} \geq \frac{x^2}{4}$. But x and z are adjacent, which contradicts Lemma 1. □

Theorem 5 *There exists a family $\{x_1, x_1+1, x_2, x_2+1\}$ of sets of 4 distinct positive integers such that the subgraph induced by these vertices in $G_+^\times$ is K_4 with one edge deleted. Namely, for every integer $c \geq 2$, we may take $x_1 = (c^2+c-1)^2 - 1$ and $x_2 = 2c^2+2c-2$.*

Proof: It is clear that x_1 and x_1+1 are adjacent and that x_2 and x_2+1 are adjacent. Also,
$x_2 = (c^2+c-1)+(c^2+c-1) = (c^2+c)+(c^2+c-2)$,
$x_2+1 = (c^2-1)+(c^2+2c)$, $x_1 = (c^2+c)(c^2+c-2) = (c^2-1)(c^2+2c)$, and
$x_1+1 = (c^2+c-1)(c^2+c-1)$.
This shows that all these vertices are pairwise adjacent, except for x_1+1 and x_2+1. Since Theorem 2 showed that there are no K_4 in $G_+^\times$ with two of the vertices being consecutive integers, x_1+1 and x_2+1 are not adjacent. □

4 Appendix

4.1 Appendix 1: Basic, Relevant Graph Theory Terms

A *simple graph* $G = (V, E)$ consists of a set V of vertices and a set E of edges, where an edge is a pair (v, w) of distinct vertices v, $w \in V$. (v, w) and (w, v) denote the same edge.

A graph G is *complete* if for each pair of distinct vertices v, $w \in G$, (v, w) is an edge in G. The complete graph on n vertices is denoted K_n.

The *induced* graph by a set of vertices $V' \subset V$ in a graph $G = (V, E)$ is the graph $G' = (V', E')$ with $E' = \{(v, w) : (v, w) \in E, \text{ and } v, w \in V'\}$.

The *chromatic number* $\chi(H)$ of a graph H is the smallest χ such that we can χ-color the vertices of H with no two adjacent vertices of the same color.

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