18.01 Problem Set 6 – Fall 2009

Due FRIDAY 11/06/09, 1:45 pm in 2-106

Part I (15 points)

Lecture 22. Friday, Oct. 30 Volumes by disks and shells.
Read: 7.4 Work: 4B-2 for parts e,g from 4B-1, 5; 4C-2,3; 4J-3

Lecture 23. Tuesday, Nov. 3 Work; average value; centroids.
Read: 7.7, to middle p. 247 Notes AV.
Work: Simmons 249/5, 6, 15 (solutions to be posted); 4D-2, 3, 5

Lecture 24. Thursday, Nov. 5 Probability; Numerical Integration.
Read: 10.9 Work: 3G-1ad, 4

Lecture 25. Friday, Nov. 6 Improper integrals, exam review.
Read: Work: Assigned on the next problem set.

EXAM 3. Tuesday, Nov. 10, covering lectures 16–25 (including differential equations)

Part II (33 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say “none” or “no consultation”. (See full explanation on PS1).

1. (Lec. 22, 3 pts) Find the volume of the solid obtained by rotating the region between the curves $y = x^2 - 3x + 2$ and $y = 0$ about the y-axis.

2. (Lec. 21/22, 5 pts) Give a formula for the volume of a donut. In mathematics, this shape is more commonly referred to as a “torus.” When setting up your integral, use R to denote the distance from the center of the hole in the donut to the center of a circular cross-section of the donut (where the creme filling is located). Also, use r to denote the radius of the circular cross-section.

3. (Lec. 23, 8 pts: 2 + 3 + 3)

a) Find the centroid (i.e. center of mass) of the lamina bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/4$.

b) Use the Theorem of Pappus to find the volume of a cone with height h and radius r. (The statement of the (First) Theorem of Pappus is on p. 391 in Simmons.)

c) Prove that the centroid of any triangle is located at the point of intersection of the medians. (Recall that the medians of the triangle are the lines connecting vertices to midpoints of the opposite side of the triangle. The medians are known to intersect each other at a distance 2/3 of the way along each median from the vertex to the opposite side.) Hint: Place vertices in the coordinate plane at $(a, 0), (0, b)$ and $(c, 0)$.

4. (Lec. 23, 4 pts)

The Mean Value Theorem for Integrals states that if f is a continuous function on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Prove the Mean Value Theorem for Integrals by applying the MVT for derivatives to the function

$$F(x) = \int_a^x f(t)\,dt.$$

5. (Lec. 24, 6 pts: 1 + 2 + 1 + 2)

a) What is the probability that $x^2 < y$ if (x, y) is chosen from the unit square $0 \le x \le 1$, $0 \le y \le 1$ with probability equal to the area.

b) What is the probability that $x^2 < y$ if (x, y) is chosen from the square $0 \le x \le 2$, $0 \le y \le 2$ with probability **proportional** to the area. (Probability = Part/Whole).

c) Evaluate

$$W = \int_0^\infty e^{-at}dt = \lim_{N\to\infty} \int_0^N e^{-at}dt$$

This is known as an improper integral because it represents the area of an unbounded region. We are using the letter W to signify "whole."

The probability that a radioactive particle will decay some time in the interval $0 \le t \le T$ is

$$P([0, T]) = \frac{\text{PART}}{\text{WHOLE}} = \frac{1}{W}\int_0^T e^{-at}dt$$

Note that $P([0, \infty)) = 1 = 100\%$.

d) The half-life is the time T for which $P([0, T]) = 1/2$. Find the value of a and W for which the half-life is $T = 1$. Suppose that a radioactive particle has a half-life of 1 second. What is the probability that it survives to time $t = 1$, but decays some time during the interval $1 \le t \le 2$? (Give an integral formula, and use a calculator to get an approximate numerical answer.)

6. (Lec. 24, 4pts) Use a calculator to make a table of values of the integrand and find approximations to the Fresnel integral $\int_0^a \cos(t^2)dt$ for $a = \sqrt{\pi/2}$, using Simpson's rule with four and eight intervals. (The exact answer to five decimal places is 1.22505. Record your approximations to six decimal places to compare.)