18.01 FALL 2009 – Problem Set 5B

Due Friday 10/30/09, 1:45 pm in 2-106

This is part B of problem set 5. The first portion of the problem set is available on the 18.01 website.

Part I (15 points)

Lecture 19. Friday, Oct. 23 First fundamental theorem of calculus.
Read: 6.6, 6.7 to top of p. 215 Work: 3C-1, 2a, 3a, 5a; 3E-6bc; 4J-2

Lecture 20. Tuesday, Oct. 27 Second fundamental theorem. Definition of log.
Read: Notes PI, p.2 [eqn.(7) and example]; Notes FT.
Work: 3E-1, 3a; 3D-1, 5, 7ab, 8a; 3E-2ac

Lecture 21. Thursday, Oct. 29 Areas between curves. Volumes by slicing.
Read: 7.1, 7.2, 7.3 Work: 4A-1a, 2, 4; 4B-1de, 6, 7

Lecture 22. Friday, Oct. 30 Volumes by disks and shells.
Read: 7.4 Work: Assigned on the next problem set

Part II (35 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. Beside each problem is the date on which corresponding material in class is covered.

0. (not until due date; 3 pts (included in total from 5A)) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Friday, 7 pts) a) Simmons 6.3/8

b) Make a conjecture about a general formula for the sum of the first n r-th powers,

$$\sum_{k=1}^{n} k^r.$$

You may not be able to conjecture an exact formula, but you should be able to describe (at least) some features of the formula, e.g. leading terms. (Hint: Use a Riemann sum interpretation of this quantity to guide you in your conjecture.)

2. (Friday/Tuesday, 3 pts) Use an integral to estimate the sum

$$\sum_{i=1}^{10{,}000} \sqrt{i}$$

3. (Friday, 4 pts) Compute

$$\lim_{x\to 3}\left(\frac{x^2}{x-3}\int_3^x \frac{\sin t}{t}dt\right).$$

4. (Friday, 16 pts: 2 + 2 + 4 + 2 + 6) Consider the function $f(x) = \int_0^x \cos(t^2)dt$. There is no expression for $f(x)$ in terms of standard elementary functions. It is known as a Fresnel integral, along with the corresponding sine integral, and appears in everything from optics (its original use) to highway design.

a) Draw a rough sketch of $\cos(t^2)$, showing the first positive and negative zeros. What does the curve look like at $t = 0$? Is the function even or odd?

b) List the critical points of $f(x)$ in the entire range $-\infty < x < \infty$. Which critical points are local maxima and which ones are local minima?

c) Sketch the graph of f on the interval $-2 \le x \le 2$, with labels for the critical points **and inflection points**. (The drawing should be qualitatively correct, but just estimate the values of f at the labelled points.)

d) Estimate $f(0.1)$ to six decimal places.

e) Fresnel integrals are sometimes expressed using different scaling of the variables. We investigate this in the following three parts.

i) Let $g(x) = \int_0^x \cos((\pi/2)u^2)du$. Make a change of variables to show that $f(x) = c_1\, g(c_2 x)$ for some constants c_1 and c_2. Why did we choose the factor $\pi/2$?

ii) Let $h(x) = \int_0^x \frac{\cos v}{\sqrt{v}}dv$. (This integral is called *improper* because $1/\sqrt{v}$ is infinite[1] at $v = 0$.) Make a different change of variable to show that $f(x) = c\, h(x^2)$ for some constant c (assume that $x > 0$).

iii) Let $k(x) = \sqrt{x}\int_0^1 \cos(xt^2)dt$, $x > 0$. Use the change of variable $z = xt^2$ and part (ii) to find the relationship between the functions k and f. Hint: Which quantities are variable and which are constant?

5. (Tuesday, 5 pts: 2 + 3)

a) Do 7.3/22.

b) Find the volume of the region in 3-space with $x > 0$, $y > 0$ and $z > 0$ given by

$$z^2/2 < x + y < z$$

Hint: First find the area of the horizontal cross-sections.

[1] Although the integrand is infinite, the area under the curve is finite. The function h is continuous, $h(0) = 0$, but its graph has infinite slope at $x = 0$.