18.01 FALL 2009 – Problem Set 3

Due Friday 10/09/09, 1:45 pm in 2-106

Part I (10 points)

Lecture 11. Fri. Oct. 2. Maximum-minimum problems.
Read: 4.3, 4.4 Work: 2C-1, 2, 4, 10, 13.

Lecture 12. Tue. Oct. 6 Related rate problems.
Read: 4.5 Work: 2E-2, 3, 5, 7

Lecture 13. Thu. Oct. 8 Newton's method.
Read: 4.6, (4.7 is optional) Work: 2F-1

Lecture 14. Fri. Oct. 9 Mean-value theorem. Inequalities.
Read: 2.6 to middle p. 77, Notes MVT Work: assigned on PS4

Part II (31 points + 10 Bonus)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation". (See full explanation on PS1).

1. (Friday, 6pts: 3 + 3) Work the following problems in Simmons' text:

a) 4.3/28 (Hint: Use as variable the distance x from the foot of the ladder to the house. Check endpoints.)

b) 4.4/28

2. (Friday, 9 pts: 3 + 3 + 3) Sketch the following functions. Your pictures should indicate asymptotes, local max and min values, intervals on which the function is increasing/decreasing and points of inflection (that is, points where the concavity changes). All this might be hard to write on the graph itself, so it would be better to include a small table with this information next to the graph.

a) $y = \dfrac{2x^2}{x^2 - 1}$

b) $y = 2\cos x + \sin 2x$

c) $y = \ln(4 - x^2)$

3. (Friday, 5 pts: 3 + 2)

a) Find the area of the largest rectangle that can be inscribed in a semicircle of radius r. In solving the problem, center the semicircle above the x-axis and express the area of the rectangle in terms of the coordinates (x, y) on the semicircle.

b) Do the problem again, but now express the coordinates in terms of an angle measure θ whose coordinates $(r\cos\theta, r\sin\theta)$ give one corner of the rectangle. (This should be easier than part (a), and shows that the way we represent the optimization problem using functions can affect the difficulty.)

4. (Thursday, 8 pts: 3 + 3 + 2) **Newton's method.**

a) Compute the cube root of 9 to 6 significant figures using Newton's method. Give the general formula, and list numerical values, starting with $x_0 = 2$. At what iteration k does the method surpass the accuracy of your calculator or computer? (Display your answers to the accuracy of your calculator or computer.)

b) For each step x_k, $k = 0, 1, \ldots$, say whether the value is i) larger or smaller than $9^{1/3}$; ii) larger or smaller than the preceding value x_{k-1}. Illustrate on the graph of $x^3 - 9$ why this is so.

c) Find a quadratic approximation to $9^{1/3}$, and estimate the difference between the quadratic approximation and the exact answer. (Hint: To get a reasonable quadratic approximation, use the fact that 8 and 9 are reasonably close.)

5. (Bonus! 10 pts: 2 + 3 + 2 + 2 + 1) **Return of the Astroid.**[1]

a) Show that every tangent line to the curve $x^{2/3} + y^{2/3} = 1$ in first quadrant has the property that portion of the line in the first quadrant has length 1. (Use implicit differentiation; this is the same as problem 45 page 114 of text.)

b) Next we reverse the logic, deriving the equation for the astroid in part (a), assuming it is a curve with the above property.

Think of the first quadrant of the xy-plane as representing the region to the right of a wall with the ground as the positive x-axis and the wall as the positive y-axis. A unit length ladder is placed vertically against the wall. The bottom of the ladder is at $x = 0$ and slides to the right along the x-axis until the ladder is horizontal. At the same time, the top of the ladder is dragged down the y-axis ending at the origin $(0, 0)$. We are going to describe the region swept out by this motion. In more picturesque language, this would be the blurry region in a photograph of the ladder's motion if the eye of the camera is open during the entire sliding process.

a) Suppose that L_1 is the line segment from $(0, y_1)$ to $(x_1, 0)$ and L_2 is the line segment from $(0, y_2)$ to $(x_2, 0)$. Find the formula for the point of intersection (x_3, y_3) of the two line segments. Don't expect the formula to be simple: It must involve all four parameters x_1, x_2, y_1, and y_2. But simplify as much as possible!

It's important to make sure you have the right formulas before proceeding further. You can doublecheck your formulas in four ways. (This is optional.)

i) If $y_2 = 0$, then $x_3 = x_1$.

ii) When the x's and y's are interchanged the formulas should be the same. What transformation of the plane does the exchange of x and y represent?

iii) It is impossible to find x_3 and y_3 if the lines are parallel, so the denominator in the formula must be zero when L_1 and L_2 have the same slope.

iv) Rescaling all variables by a factor c leaves the formula unchanged, so the numerator of the formula for x_3 and y_3 should have degree (in all variables) one greater than the denominator.

b) Write the equation involving x_2 and y_2 that expresses the property that ladder L_2 has length

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[1] Bonus problems are completely optional. Your bonus points are recorded in a separate column in Stellar to avoid affecting the PSet average.

one. We will suppose that L_1 represents the ladder at a fixed position, and L_2 tends to L_1. Thus

$$x_2 = x_1 + \Delta x; \quad y_2 = y_1 + \Delta y$$

Use implicit differentiation (related rates) to find

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(Express the limit as a function of the fixed values x_1 and y_1.)

c) Substitute $x_2 = x_1 + \Delta x$ and $y_2 = y_1 + \Delta y$ into the formula in part (a) for x_3 and use part (b) to compute

$$X = \lim_{x_2 \to x_1} x_3 = \lim_{\Delta x \to 0} x_3$$

Simplify as much as possible. Deduce, by symmetry alone, the formula for

$$Y = \lim_{x_2 \to x_1} y_3$$

d) Show that $X^{2/3} + Y^{2/3} = 1$. (The limit point (X, Y) that you found in part (c) is expressed as a function of x_1 and y_1. This is the unique point of the ladder L_1 that is also part of the boundary curve of the region swept out by the family of ladders.)