Worksheet 3: Limits and Derivatives

18.01 Fall 2009

Problem 1. Find the derivative of $\cos x$ from the definition of the derivative. You will probably need to use the trigonometric limits mentioned in lecture.

$$\lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0} \frac{\cos x \cosh - \sin x \sinh - \cancel{\cos x}}{h} = \lim_{h\to 0}\left(\frac{\cos x\,(\cos h-1)}{h} - \sin x \frac{\sin h}{h}\right)$$

$= \cos^2 h - 1 = \sin^2 h$ 1 in lim   1 in lim

$$= \lim_{h\to 0}\left(\cos x \frac{(\cos h-1)(\cos h+1)}{h\,(\cos h+1)} - \sin x \frac{\sin h}{h}\right) = \lim_{h\to 0}\left((\cos x)\left(\frac{\sin h}{h}\right)\left(\frac{\sin h}{\cos h+1}\right) - \sin x \frac{\sin h}{h}\right) =$$

$$= \cos x \cdot 1 \cdot \frac{0}{2} - \sin x \cdot 1 = \boxed{-\sin x}$$

Problem 2. Compute the following limits

a) $\lim_{x\to 0} \frac{x^2}{\sin x}$ $= \lim_{x\to 0} x \cdot \frac{x}{\sin x} = 0 \cdot 1 = 0.$

b) $\lim_{x\to 5}(5y+3x+2)$ $= 5y + 3\cdot 5 + 2 = 5y+17.$

c) $\lim_{s\to 0} \frac{\cos(\pi/2+s)-\cos(\pi/2)}{s}$ $= \frac{d}{dx}(\cos x)\Big|_{x=\pi/2} = -\sin x\Big|_{x=\pi/2} = -\sin(\pi/2) = \boxed{-1}$ //.

d) $\lim_{x\to\pi/3} \frac{\cos(x)-0.5}{x-\pi/3}$ $= \lim_{x\to\pi/3} \frac{\cos x - \cos\pi/3}{x-\pi/3} = \frac{d}{dx}(\cos x)\Big|_{x=\pi/3} = -\sin\pi/3 = \boxed{-\frac{\sqrt{3}}{2}}$ //

$\cos\pi/3 = 1/2$

Problem 3. Suppose $f'(3) = 5$, $g'(3) = 2$, and $f(3) = g(3) = 2$. Compute

a) $(fg)'(3)$ $= f'(3)g(3) + f(3)g'(3) = 5\cdot 2 + 2\cdot 2 = 10+4 = 14.$

b) $(f/g)'(3)$ $\frac{f'(3)g(3) - f(3)g'(3)}{(g(3))^2} = \frac{5\cdot 2 - 2\cdot 2}{(2)^2} = \frac{10-4}{4} = \frac{6}{4} = \frac{3}{2}$ //.

c) $(3f - g'(3) * g)'(3)$

$3f'(3) - g'(3)\, g'(3) = 3\cdot 5 - \cancel{4}2^2 = 15-4 = 11.$