18.01: REVIEW FOR EXAM 3

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Contents

1. Differential equations

The equation here can be written in the form $y' = \frac{f(x)}{g(y)}$. Rewrite the equation as $g(y)dy = f(x)dx$. Let $F(x), G(y)$ be antiderivatives of $f(x), g(y)$. Then we get $F(x) - G(y) = C$, where C is some constant. So we can express y in terms of x. The expression will still include an unknown constant C. This constant is usually recovered from knowing the value of $y(x)$ at some fixed point a.

Problem to practice (from exam 2)

- Practice questions for exam 2, problem 10.
- Practice exam 2, problem 5b.
- Exam 2, problem 5b.

2. Definite integrals vs Riemann sums

2.1. Riemann sums from definite integrals.

Let $f(x)$ be a function defined on an interval $[a, b]$. Pick an integer n and divide $[a, b]$ into n equal intervals (each of length $\frac{b-a}{n}$). These intervals will be $[a, a+\frac{b-a}{n}], [a+\frac{b-a}{n}, a+2\frac{b-a}{n}], \ldots, [a+(i-1)\frac{b-a}{n}, a+i\frac{b-a}{n}], \ldots, [a+(n-1)\frac{b-a}{n}, b]$. The interval number i has end-points $a + (i-1)\frac{b-a}{n}, a + i\frac{b-a}{n}$. This is because there are $i-1$ intervals of length $\frac{b-a}{n}$ before the left-end point and i such intervals before the right end-point.

Pick points $x_1, \ldots, x_n$, where x_i lies on the interval number i, so that $a + (i-1)\frac{b-a}{n} \leqslant x_i \leqslant a + i\frac{b-a}{n}$. Then

$$\frac{b-a}{n}\sum_{i=1}^{n} f(x_i) = \frac{b-a}{n}(f(x_1) + f(x_2) + \ldots + f(x_n))$$

is a Riemann sum for the integral $\int_a^b f(x)dx$. If $f(x)$ is continuous, then (by definition) the integral $\int_a^b f(x)dx$ is the limit of its Riemann sums as $n \to +\infty$.

There are some distinguished Riemann sums:

- Left R.S.: here x_i is the left end-point of the i-th interval, i.e., $x_i := a + (i-1)\frac{b-a}{n}$.
- Right R.S.: here x_i is the right end-point of the i-th interval, i.e., $x_i := a + i\frac{b-a}{n}$.
- Upper R.S.: here $f(x_i)$ is the maximal value of $f(x)$ on the i-th interval.
- Lower R.S.: here $f(x_i)$ is the minimal value of $f(x)$ on the i-th interval.

In particular, the integral $\int_a^b f(x)dx$ is always $\geqslant$ any of its lower R.S., but is always $\leqslant$ any of its upper Riemann sum.

- Practice questions, problem 3.
- Exam, problem 2a.

2.2. Definite integrals from Riemann sums.

The goal here is to recover integrals from their Riemann sums. A typical problem is to compute a certain limit (or to express it as an integral).

A guide to recognizing Riemann sums.

A problem here looks like the following: Compute the limit

$$\lim_{n\to\infty} \frac{d}{n}(F_1 + \ldots + F_n).$$

Here d will be some number, and $F_1, \ldots, F_n$ will be values of an appropriate function $f(x)$ at some points $x_1, \ldots, x_n$ (in many cases a function should be clear from the formula).

This limit is calculated by relating it to the integral $\int_a^b f(x)dx$. Here:

- f is a function mentioned above.
- We mostly have left or right Riemann sums in these problems. In the case of a left sum, the smallest argument x_1 is the same for all n, it will coincide with the left end point a. To determine the right end-point b one can consider the largest argument x_n. It will depend on n but will approach b as $n \to +\infty$. The points $x_1, \ldots, x_n$ will form an arithmetic series with difference $\frac{b-a}{n}$.

  In the case of a right sum, the largest argument x_n is the same for all n. This common value will be b. To determine a compute the limit of the smallest argument x_1. The points $x_1, \ldots, x_n$ form an arithmetic series with difference $\frac{b-a}{n}$.

  In general, a is still the limit of x_1, and b is the limit of x_n.

2.3. Computation of an integral from definition.

The strategy here is as follows:

1. Write down the required Riemann sum (if no requirement is made, left or right should be used).
2. Compute the Riemann sum for given n.
3. Compute the limit as $n \to \infty$.

- Practice questions, problem 2.

3. Fundamental theorems of calculus

3.1. **Statements.**
1st (form of) FTC: If $f(x)$ is continuous, then $\frac{d}{dx}\int_a^x f(t)dt = f(x)$ (please note that the variable of integration is different from x).

2nd (form of) FTC: If F(x) is antiderivative of f(x), then $\int_a^b f(t)dt = F(b) - F(a)$.

FTC (especially in its second form) is a powerful tool to compute definite integrals.

3.2. **Integration via substitution.**
Recall that if $u = u(x)$, then $\int f(u(x))u'(x)dx = \int f(u)du$ (follows from the chain rule). Then one can computes $\int f(u)du$ and plugs $u = u(x)$ in the result.

When we compute definite integrals we can make substitution both for the integrand and for the bounds: $\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$.

This is convenient because we do not need to remember which substitution we made, we can just compute the integral on the right-hand side.

Problems to practice:

- Practice questions, problem 1.
- Exam 3, problem 1, problem 5d.

3.3. **Derivatives of integrals.**
Problems here are about computation of derivatives like $\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt$ (we remark that the function we differentiate is a function of x not of t). In easier problems $a(x)$ will be constant and $b(x)$ will be x, then the answer is just $f(x)$ (by the 1st FTC).

In more difficult problems both $a(x)$ and $b(x)$ do depend on x. One way to do such problems is, first, to compute the integral explicitly and then to differentiate it. However, this requires too much work. Actually, there is a general formula that is proved by using the 2nd FTC and the chain rule:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))b'(x) - f(a(x))a'(x).$$

Problems to practice.

- Practice questions, problem 4b), problem 5.
- Practice exam: problem 3 a),b).
- Exam, problem 5 a),b).

4. Applications: areas,volumes etc.

4.1. **Areas.**
To find an area of some region one needs

- to place it to a coordinate plane (if it is not there from the beginning).
- Compute the length $l(x)$ the cross-section of the region by the vertical line corresponding to x.
- Determine bounds $a < b$. Usually they are specified in the statement of the problem or are such that the region is enclosed btw. $x = a$ and $x = b$.

For instance, a possible problem is to determine the area between the graphs $y = f(x), y = g(x)$. The simplest possibility here is that the graphs have only two points of intersection: $x = a$ and $x = b$ with $a < b$. If $f(x) \geqslant g(x)$ for $x \in (a, b)$, then the area is given by $A = \int_a^b f(x)dx$. For three points of intersection ($a < b < c$) the formula becomes more complicated: $A = \int_a^b |f(x) - g(x)|dx + \int_b^c |f(x) - g(x)|dx$, etc.

Sometimes, it is more convenient to use horizontal lines to form the cross-section. The formula for the area will look line $\int_a^b l(y)dy$. The basic principle to choose between the two choices of coordinates is that one needs to use cross-sections producing easier/more explicit integrals.

4.2. **General volumes.** To find a volume of some region one needs

- to place it to a coordinate space (if it is not there from the beginning).
- Compute the area $A(x)$ of the cross-section of the solid by a plane consisting of all points with given x-coordinate (equal to x). Instead of x one can take either of the other two coordinates.
- Determine bounds $a < b$ (with the same reasoning as for areas).

Here problems of two type are possible. Either the solid is given by some inequalities (like in PSet 5b, problem 5) or it is given geometrically (and not placed into the coordinate space). In the first case, cross-sections will be again given by inequalities (with, say, x fixed) and bounds a, b will be such that the inequalities have no solutions for $x < a$ and $x > b$. In the second case one needs to choose direction of cross-sections so that they become as simple geometrically as possible.

4.3. **Volumes with rotational symmetries.** When a solid has rotational symmetry (= obtained by revolving some flat region about a line) there are special methods of computing its volume: disks (washers) and shells.

A disk method is a general method described above, where one takes cross-sections perpendicular to axis of rotation. In this case a cross-section is a disk or, more generally, a "washer" (a disk where a smaller disk with the same center is excluded).

A shell method is different. We cut the solid into thin cylindrical shells. The area of the corresponding cross-section is $2\pi r h(r)$, where r is its radius (=distance from the cross-section to the axis of rotation) and $h(r)$ is the height of the corresponding shell.

For instance, consider the following problem. Consider the region given by $a \leqslant x \leqslant b, g(x) \leqslant y \leqslant f(x)$ (we assume that $0 \leqslant a$ and $0 \leqslant g(x)$ for $x \in [a, b]$). Then we rotate the region about one of the coordinate axis.

If we rotate it about x-axis, it is better to use the disk method because it produces the integral very easy. Namely, cross-section is a washer with smaller radius $g(x)$ and larger radius $f(x)$. So the volume will be $\int_a^b \pi(f(x)^2 - g(x)^2)dx$.

If we rotate our region about y-axis, it is better to use the shell method. The height of the shell at position x is $f(x) - g(x)$. So the volume is $\int_a^b 2\pi x(f(x) - g(x))dx$.

Another trick that can be used to compute volumes is the Pappus theorem, see the section on centroids.

Problems to practice:

- Practice questions: Problems 6,7.
- Practice exam: Problem 2.
- Exam: Problem 3.

4.4. **Average value.** The average value of the function $f(x)$ on an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)dx$.

Problems to practice:

- Practice questions, problem 8.
- Practice exam, problem 5.
- Exam 4b (although this is not a typical problem here).

4.5. **Centroids.** Given a region on the plain, its center of mass is the point with coordinates $(m_y/m, m_x/m)$, where m is the area of the region and m_y, m_x are momenta about y- and x-axis computed as follows. Let $h(x)$ ("height") be the length of the cross-section of the region by the vertical line with x-coordinate equal to x. Similarly, let $w(y)$ ("width") be the length of the cross-section of the region by the vertical line with y-coordinate equal to y. Then

$$m_y = \int_a^b xh(x)dx, m_x = \int_c^d yw(y)dy, m = \int_a^b h(x)dx = \int_c^d w(y)dy.$$

The center of mass (centroid) is easier to determine when a region has some symmetry. For instance, if it is symmetric about a point, then its centroid is just this point. If a region is symmetric about some line, then its centroid lies on this line.

Centroids have a nice application to computation of volumes (the 1st Pappus theorem). Namely, given a region on a plane and some line such that the region lies entirely on one side from the line, consider the solid obtained by revolution of the region about the line. Its volume V is equal to $2\pi Ar$, where A is the area of the region and r is the distance from the centroid to the axis of rotation. This is especially helpful when we know the centroid without computing integrals, i.e., when the region is symmetric about a point, or when we just know the distance r. This happens when the region is symmetric about the line that is parallel to the axis of revolution (the centroid just lies on that line).

Also the Pappus theorem may be useful when the line of revolution is not parallel to coordinate axis.

Remark. Sometimes computing m_y, m_x by the formulas given above is not practical. There are alternative formulas for m_y and m_x. They can be deduced from the Pappus theorem. Namely, the volume of the solid obtained by rotating the region about x-axis is $2\pi m_x$ (the computation using the shell method). On the other hand, we can compute the same volume using the disk method, which will give a different (sometimes easier) integral.

For example, suppose that our region is given by $a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)$. Then we have $m_y = \int_a^b xf(x)dx, m_x = \int_a^b \frac{f(x)^2}{2}dx$.

4.6. **Probability.** Let x be some quantity. We want to determine how often x takes values btw. a and b, or, in other words, the *probability* of the event $x \in [a, b]$. In some cases to answer this question we need to know a probability distribution $f(x)$.

A *probability distribution* $f(x)$ is a function with $f(x) \geqslant 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$. The probability of the event $x \in [a, b]$ with distribution function $f(x)$ is $\int_a^b f(x)dx$. Sometimes, the domain of all possible values of x is not the whole $(-\infty, +\infty)$, for instance, x may take only positive values. Then we consider $f(x)$ with $\int_0^{\infty} f(x)dx$.

Now suppose we want to determine the average value of some quantity y depending on x (the term for this average value is "expectation" or "mathematical expectation"). Then it is given by the integral $\int_{-\infty}^{\infty} y(x)f(x)dx$. If x takes values only on $[A, B]$, then the integral becomes $\int_A^B y(x)f(x)dx$.

Other problems on probability are possible, compare with PSet6, problem 5.

Problems to practice: Exam, problem 4.

5. NUMERICAL INTEGRATION

This includes methods for computing approximate values of integrals. All methods we have are ramifications of Riemann sums. More precisely, we approximate an integral on a small interval using some values of a function on this interval. Then, for large intervals, we break them into n small intervals, make an approximation on each small interval, and then add all n approximations.

Basic formulas (approximations on a single interval).

Trapezoidal rule. $\int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2}$ (when $f(a), f(b) > 0$ this is just the area of the trapeze with vertices $(a, f(a)), (a, 0), (b, 0), (b, f(b))$).

Simpson rule. $\int_a^b f(x)dx \approx (b-a)(\frac{1}{6}f(a) + \frac{4}{6}f(\frac{a+b}{2}) + \frac{1}{6}f(b))$. Geometrical meaning of this formula is the following. Draw a parabola through the points $(a, f(a)), (\frac{a+b}{2}, f(\frac{a+b}{2})), (b, f(b))$. If $f(a), f(\frac{a+b}{2}), f(b) > 0$, then the r.h.s. of the previous approximate equality is the area under this parabola, for $a \leqslant x \leqslant b$.

In other words, the coefficients of $f(a), f(\frac{a+b}{2}), f(b)$ are chosen such that the approximation is exact for $1, x, x^2$ (as it happens, due to symmetry, the formula also gives an exact value for $f(x) = x^3$).

General formulas.

Trapezoidal rule. Divide an interval $[a, b]$ into $\underline{n}$ small intervals (of length $\frac{b-a}{n}$). Let $x_i, i = 0, 1, \ldots, n$, be end-points, $x_i = a + i\frac{b-a}{n}$. Set $y_i = f(x_i)$. Then the approximation is given by

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\sum_{i=0}^{n-1}\frac{y_i + y_{i+1}}{2} = \frac{b-a}{n}\left(\frac{1}{2}y_0 + y_1 + \ldots + y_{n-1} + \frac{1}{2}y_n\right).$$

Simpson rule. Divide an interval $[a, b]$ into $\underline{2n}$ small intervals (of length $\frac{b-a}{2n}$). Let $x_i, i = 0, 1, \ldots, 2n$, be end-points, $x_i = a + i\frac{b-a}{2n}$. Set $y_i = f(x_i)$. Then the approximation is given by

$$\begin{aligned}\int_a^b f(x)dx \approx& \frac{b-a}{n}\sum_{i=0}^{n-1}\frac{y_{2i} + 4y_{2i+1} + y_{2i+2}}{6}\\ &= \frac{b-a}{n}(\frac{1}{6}y_0 + \frac{4}{6}y_1 + \frac{2}{6}y_2 + \frac{4}{6}y_3 + \frac{2}{6}y_4 + \ldots + \frac{2}{6}y_{2n-2} + \frac{4}{6}y_{2n-1} + \frac{1}{6}y_{2n}).\end{aligned}$$

Problems to practice.

- Practice question, problem 9.
- Practice exam, problem 5.
- Exam, problem 2b,c.