18.01: REVIEW FOR EXAM 2

IVAN LOSEV

Contents

1. Approximation

Approximation of $f(x)$ at (near) $x = x_0$
Linear: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.
Quadratic: $f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$.

Standard approximations:
The following approximations **are at** $x = 0$.

Table 1: Basic approximations at $x = 0$

Function $f(x)$	Linear appr.	Quadratic appr.
e^x	$1 + x$	$1 + x + \frac{1}{2}x^2$
$\sin(x)$	x	x
$\cos(x)$	1	$1 - \frac{1}{2}x^2$
$\ln(1 + x)$	x	$x - \frac{1}{2}x^2$
$(1 + x)^r$	$1 + rx$	$1 + rx + \frac{r(r-1)}{2}x^2$

Problems to practice[1]**:**

- Practice questions, Problem 7.
- Practice exam, Problem 5a.
- Exam, Problem 1.

Remark. Let us explain a recipe for getting the formulas for approximation. For instance, the linear (or, the other name, 1st order) approximation $f(x_0) + f'(x_0)(x - x_0)$ (denote it by $l(x)$) for $f(x)$ at $x = x_0$ is the only linear function such that $l(x_0) = f(x_0), l'(x_0) = f'(x_0)$.

[1]All problems are taken from exam 2 materials on OCW

Similarly, the quadratic (2nd order) approximation (say, $q(x)$) is the only quadratic function such that $q(x_0) = f(x_0), q'(x_0) = f'(x_0), q''(x_0) = f''(x_0)$.

2. Graphing

Graphing of a function includes:

1. Finding critical points ($f'(x) = 0$), intervals, where a function is increasing ($f'(x) > 0$) and decreasing ($f'(x) < 0$).

2. Finding inflection points ($f''(x) = 0$) and intervals, where a function is concave up ($f''(x) > 0$) and down ($f''(x) < 0$).

3. Finding vertical asymptotes, if any. These are vertical lines $x = a$, where $f(x)$ is undefined at $x = a$ and, moreover, one or both of single sided limits $\lim_{x\to a^+} f(x), \lim_{x\to a^-} f(x)$ is $+\infty$ of $-\infty$.

4. Finding horizontal asymptotes. This are horizontal lines $y = b$ such that one (or two) of limits $\lim_{x\to+\infty} f(x), \lim_{x\to-\infty} f(x)$ exists and equals b.

5. Finding slant asymptotes (a more advanced stuff). This are lines $y = ax + b$ such that $\lim_{x\to+\infty}(f(x) - ax - b) = 0$ or $\lim_{x\to-\infty}(f(x) - ax - b) = 0$. The first equality, for example, means that the graph of $y = f(x)$ approaches $y = ax + b$ as $x \to +\infty$ (example: look at $f(x) = x + 1 + \frac{1}{x}$; here $y = x + 1$ is a slant asymptote).

6. Symmetries (even/odd) and/or periodicity.

7. Finally, it is useful to find a few points on the graph, in particular, to find zeroes of a function.

Problems to practice:

- Practice questions: problems 1,2.
- Practice exam: problem 1.
- Exam: problem 1.

Remark. Sometimes a problem explicitly says which steps it wants from you.

3. Max-min problems

Max-min problems can appear in two different forms: "explicit" and "implicit".

"Explicit": Here we are given a function $f(x)$ and asked to find a point of min/max (and/or compute min/max value) on a "closed" interval $[a, b]$ (meaning that $a \leqslant x \leqslant b$) or on an "open" interval (a, b) (meaning that $a < x < b$, here it is possible that, for instance, $a = +\infty$) etc.

"Implicit": Here we don't have any function from the beginning. Instead we need to min/max some geometric, physical etc. quantity. The usual way to deal with such problems is to reduce them to explicit ones and then solve the latter.

How to reduce an "implicit problem" to an "explicit" one.

One can proceed in the following three steps that are illustrated below by an example.

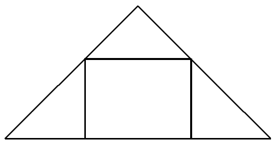
Step 1. Pick a variable.

Step 2. Write down a function (of that variable) to be maximized/minimized.

Step 3. Determine the domain, where the function should be maximized/minimized, i.e., all possible values of the variable.

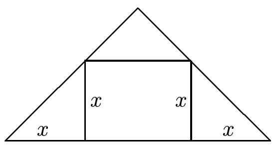
Example. *Suppl. notes, 2C-8 b): Find the dimensions of the rectangle of largest area inscribed in the right triangle so that one side of the rectangle is parallel to the hypotenuse. We assume for simplicity that the catheti of the triangle are equal.*

Being geometric, this problem has "Step 0": draw a picture



Step 1. We asked to find dimensions. Therefore it is natural to take one of the dimensions of the rectangle as a variable. It is a bit more convenient to denote the vertical dimension by x. Also we need the notation for one of dimensions of the triangle: denote the length of cathetus by a (this is not a variable, it is fixed).

Step 2. We need to express the area of the rectangle in terms of x. The area equals to the product of the vertical and horizontal dimensions, so we need to express the horizontal dimension in terms of x. To do this look at the following picture:



Since the length of the hypotenuse is $\sqrt{2}a$, we see that the horizontal dimension of the rectangle is $\sqrt{2}a - 2x$. So the area equals $x(\sqrt{2}a - 2x) = \sqrt{2}ax - 2x^2$.

Step 3. We need to understand the domain of possible values of x. Clearly, $x \geqslant 0$ (for $x = 0$ the rectangle "degenerates" to a horizontal line). Also x cannot be bigger than the height of the triangle, which is $\sqrt{2}a/2$ (if the equality holds, then the rectangle degenerates to a vertical line). So $x \in [0, \sqrt{2}a/2]$.

So we arrive to the following problem: *find* $x \in [0, \sqrt{2}a/2]$ maximizing $f(x) = \sqrt{2}ax - 2x^2$.

How to solve an "explicit" problem.

Suppose we want to maximize/minimize a function $f(x)$ on the interval $[a, b]$.

Step 1. Find all critical points of $f(x)$ lying on (a, b).

Step 2. Here we have several options. The most straightforward one is as follows:

we know that the maximal/minimal value is achieved either in a critical point of in an endpoint. So one can compare values in these points and choose the maximal/minimal one.

Sometimes, however, this computation is not practical (we have many critical points, or it is difficult to compare values). Then one can solve try to solve the problem based on the computation of the second derivative. There are several observations to be made:

(1) if $f''(x) > 0$ for a critical point x, then x is a point of local minimum. Therefore x cannot be a point of global maximum. In some situations, x is not a point of global minimum (even if it is a unique critical point of local minimum – try to produce a picture, one should have at least one point of local maximum).

(2) If there is a unique critical point (as it happens in many geometric problems), then the situation is much simpler. Namely, let x be a point of local minimum, and there are no other critical points on the interval in consideration. Then x is the point of global minimum, and the maximum is achieved in one of the critical points. The situation when x is a point of local maximum is similar[2].

Example. Return to the problem we considered: maximize the function $f(x) = \sqrt{2}ax - 2x^2$ for $x \in [0, \sqrt{2}a/2]$.

[2] What happens when x is critical but neither local minimum nor maximum?

Step 1. Find critical points: $f'(x) = \sqrt{2}a - 4x$. So there is a unique critical point $x = \sqrt{2}a/4$.

Step 2. The value of $f(x)$ at both end points is zero. So $x = \sqrt{2}a/4$ does maximize the function. The maximal value of $f(x)$ is $f(\sqrt{2}a/4) = \sqrt{2}a\sqrt{2}a/4 - 2(\sqrt{2}a/4)^2 = a^2/4$. Also we can argue as in observation (2) above: $f''(x) = -4$, so $\sqrt{2}a/4$ is a point of local maximum. Being a unique critical point, it is the point of global maximum.

Problems to practice:

- Practice questions: Problems 3 and 4.
- Practice exam: Problem 2.
- Exam: Problem 3.

Remark (what happens if we are dealing with the open interval (a, b) instead of the closed interval $[a, b]$). In this case we replace the values $f(a), f(b)$ with limits $\lim_{x\to a} f(x), \lim_{x\to b} f(x)$. However, if it happens that one of this values is, say, smaller than all values at critical points, then the function $f(x)$ does not achieve a minimum on (a, b). For instance, there is no x maximizing $f(x) = x^2$ on the interval (-1,1). Indeed, for any $x \in (-1, 1)$ there is another point $y \in (-1, 1)$ with $f(y) > f(x)$ (just take y with $|y| > |x|$).

4. Related rates

In a related rates problem we have two quantities depending on time but subject to a relation that does not depend on time. One needs to express the rate of change of one quantity (at a specified moment) in terms of that for the other.

In many (if not most) cases it is reasonable to approach a related rates problem using the following steps[3].

Step 1. Introduce (notation for) quantities. If you have more than 2 possible quantities (e.g., dimensions of a right triangle) you can use the following rule: your first quantity will be that with known rate of change, say $x(t)$, while your second quantity will be one whose rate of change you want to determine, say $y(t)$.

Step 2. Read carefully the statement to gather all information contained there (this may be tricky). In many situations this information will involve an unknown moment of time t_0. Also write down what you need to find.

Step 3. Write down a relation btw. $x(t), y(t)$. In many problems this will have a geometric origin. In other situations (say, physical) this relation can be (intrinsically) a part of the statement.

Step 4. Differentiate the relation. Express $y'(t)$ in terms of everything else.

Step 5. Plug all information you have. In most cases, the answer should be a number. If you do not get a number (your answer involves some unknown values), it may mean that you missed something on step 2.

Example.

A 200 foot tree is falling in the forest; the sun is directly overhead. At the moment when the tree makes an angle of 30° with the horizontal, its shadow is lengthening at the rate of 50 feet/sec. How fast is the angle changing at that moment?

Solution.

[3] please note that this recipe does not always give the easiest solution, see, for example, 2E-3 from suppl. notes

Step 1. The quantity with known rate of change is the position of the projection of the tree at the moment t, say $x(t)$. The quantity, whose rate of change we want to find is the angle, say $\theta(t)$.

Step 2. We know the height of the tree: 200 feet. We also know that at the certain moment t_0, the angle $\theta(t_0) = \pi/6 = 30^\circ$ and $x'(t_0) = 50$(feet/sec). What we want to find is $\theta'(t_0)$.

Step 3. The relation comes from geometry: $x(t)$ is the cathetus in the right triangle, whose hypotenuse is 200 and the angle between the cathetus and the hypotenuse is $\theta(t)$. So $x(t) = 200\cos\theta(t)$.

Step 4. $x'(t) = -200\sin(\theta(t))\theta'(t)$ and therefore $\theta'(t) = -\frac{x'(t)}{200\sin(\theta(t))}$.

Step 5. Plugging $\theta(t_0) = \pi/6, x'(t_0) = 50$, we get $\theta'(t_0) = -\frac{50}{200\cdot 0.5} = -0.5$(rad/sec). Please note the unit.

Problems to practice:

- Practice questions: problems 4 and 5.
- Practice exam: problem 4.
- Exam: problem 4 (partly).

5. Newton method

This is a method to find approximately solutions of the equation $f(x) = 0$, where $f(x)$ is a differentiable function. This method produces a solution by iterations. More precisely, we construct a sequence x_n of points. Under favorable circumstances, this sequence approaches the actual solution x.

We start by picking a point x_0 (in practice, this point should be "close" to a solution we expect to find). The next point x_1 is obtained by the formula: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ (draw the tangent to $y = f(x)$ at the point $(x_0, f(x_0))$, then x_1 is the point of intersection of this tangent with the x-axis). In general, having constructed a point x_n, we set

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}.$$

Remark. The natural question here is: what are "favorable circumstances"? There are several possible answers. One of them is as follows:

Suppose that $f(x)$ has a zero (otherwise, there is nothing to find), and that $f(x)$ is increasing and concave up (decreasing and concave up, etc., also works). Then Newton method always works no matter which x_0 you choose (however, you still would like to choose a point close to the actual solution to make the method work faster). Moreover, if you draw a picture, you can notice the following pattern: x_1 is always bigger, than the actual solution, and each x_{n+1} lies between x_n and the actual solution.

6. Mean Value Theorem

The theorem asserts the following:

- if $f(x)$ is a differentiable function and $a < b$ are numbers, then there is a point c with $a < c < b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

6.1. **MVT and inequalities.** A typical problem here is to show that $f(x) \geqslant 0$ for all $x \geqslant a$, where a is some fixed number (variant: $f(x) < 0$ for $x > a$; or $f(x) \geqslant 0$ for $x \leqslant a$, etc.).

Usually these problems are solved as follows (we consider the problem with $f(x) < 0$ for $x > a$):

It is often happens that $f(a) = 0$ (although $f(a) < 0$ is also OK). We need to show that $f(b) < 0$ for all $b > a$. This will follow if we show that $f(b) - f(a) < 0$ for all $b > a$, or, equivalently, that $\frac{f(b)-f(a)}{b-a} < 0$. By MVT, $\frac{f(b)-f(a)}{b-a} = f'(c)$ for some $c \in (a, b)$. So our problem reduces to checking $f'(x) < 0$ for all $x > a$. This is often doable.

Useful exercise. Make sure you can modify the argument in the previous paragraph to make it work for the problem like: show that $f(x) > 0$ for all $x < a$.

Example.

Show that $\sin(x) < x$ *for all* $x > 0$.

Rewrite the inequality in interest as $\sin(x) - x < 0$. Our $f(x)$ is $\sin(x) - x$. We need to make sure that $f'(x) < 0$ for all $x > 0$. But $f'(x) = \cos(x) - 1$. Since $\cos(x) \leqslant 1$ for all x, we see that $f'(x) \leqslant 0$. It follows that $f(x) < 0$ for all $x > 0$, and we are done[4].

Problems to practice:

- Practice questions: problem 8.
- Practice exam: problem 6, part a.

6.2. **Other applications.** We also have some other applications of MVT, see Practice exam, problem 6b, or Exam, Problem 6.

[4]There is a little mistake in this argument (on exam, this may be worth 1 point, especially if a grader has headache/ problems with digestion, etc.) Actually, we have proved that $f'(x) \leqslant 0$, although in the general argument above we should have had $f'(x) < 0$. However, since $f'(x) = 0$ only in isolated points ($x = 2\pi k$, where k is integer) this does not matter.