18.01: REVIEW FOR EXAM 1

IVAN LOSEV

Contents

1. Computation of derivatives

1.1. Basic derivatives.
These are to be memorized, although some of them can be deduced from others using derivation rules, see below.

Table 1: Basic derivatives

no	Function $f(x)$	Derivative $f'(x)$
1	x^a	ax^{a-1}
2	$\sin(x)$	$\cos(x)$
3	$\cos(x)$	$-\sin(x)$
4	e^x	e^x
5	$\ln(x)$	$1/x$

A useful generalization of 4th row: $\frac{d}{dx}a^x = \ln(a)a^x$.

1.2. Derivation rules.
These are roughly divided into two groups:

Differentiation vs algebra.
Sum rule: $(f(x)+g(x))' = f'(x)+g'(x)$.

Product rule: $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$.

Quotient rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$.

A useful special case of the quotient rule: $\left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{g(x)^2}$.

Problems to practice:

- preexam 1a, problem 1a;
- preexam 1b, problem 1;
- suppl. notes 1E.

Differentiation vs compositions of functions.
Chain rule: $(f(g(x)))' = f'(g(x))g'(x)$. This is a basic rule here.

Inverse rule: $(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$.

Problems to practice:

- preexam 1a, problems 1d,4;
- preexam 1b, problem 2;
- Suppl. Notes 1G;

Implicit differentiation: This is another technique related to the chain rule.
A function $y = y(x)$ is given implictly by the equality $F(x, y) = 0$ (e.g., $x^2 + y^2 - 1 = 0$ or $x^2y - xy^3 + 6 = 0$). Then to find the derivative $y'(x_0)$ we:
a. Differentiate $F(x, y(x))$ using the chain rule. E.g. in the second example we get:

$$2xy + x^2y' - y^3 - x \cdot 3y^2y' = 0.$$

b. Express y' in terms of x and y, e.g.,

$$y' = \frac{2xy - y^3}{3xy^2 - x^2}.$$

Then depending on the situation we do one of the following:
c1) Plug the values of x and y for which we want to compute the derivative. In the previous example for $x_0 = 1, y_0 = 2$ we get $y'(x_0) = -\frac{4}{11}$. We do this, for example, if asked to find the equation of the tangent line to $F(x, y) = 0$ at the point (x_0, y_0) on the graph (i.e., $F(x_0, y_0) = 0$).
c2) Or we can try to express y in terms of x from $F(x, y) = 0$. This will allow us to express y' in terms of x.
We also remark that it is possible that the value of y isn't determined uniquely from the value of x. E.g., for $x^2 + y^2 - 1 = 0$ for any x with $-1 < x < 1$ there are two possible values of y: $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$.

Problems to practice:

- preexam 1b, problem 4;
- preexam 1a, problem 6 (related to tangent lines);
- Suppl. notes 1G;

2. Computation of limits

2.1. Computation of limits without using derivatives.

Limits vs algebra. Limits are "compatible" with algebraic operations. That is, if $\lim_{x\to x_0} f(x) = A, \lim_{x\to x_0} g(x) = B$, then

1. $\lim_{x\to x_0} f(x) + g(x) = A + B$.
2. $\lim_{x\to x_0} f(x)g(x) = AB$.
3. $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}$ provided $B \neq 0$.

The computation of limits $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ with $A = B = 0$ requires additional considerations, for example, one can try to relate such limits to derivatives.

Trigonometric limits.

The most basic limit here: $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Most trigonometric limits are deduced from this one with the possible exceptions of limits related to derivatives.

Problems to practice (all limits without derivatives)

- Preexam 1a, problem 6a.
- Preexam 1b, problem 1b.
- Suppl. notes 1D.

2.2. Limits vs derivatives. Limits vs derivatives.

Here two types of problems are possible:

Limits via derivatives. Uses the definition of the derivative: $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$.

Problems to practice:

- Preexam 1a, problem 3.
- Preexam 1b, problem 6b.

Derivatives via limits. These are problems asking to compute the derivative directly from definition.

Problems to practice:

- Preexam 1a, problem 2.
- Preexam 1b, problem 3.
- Wsh 3, problem 1.

3. Functions and their graphs

3.1. Tangent lines.

Case 1. Function is given explicitly: $y = f(x)$. Then the equation of the tangent line to the point $(x_0, y_0), y_0 = f(x_0)$ (we remark that a point has to lie on the curve) is:

$$y = f'(x_0)(x - x_0) + y_0 = f'(x_0)x + (y_0 - x_0 f(x_0)).$$

Case 2. A function $y = y(x)$ is given implicitly by the equation $F(x, y) = 0$ (e.g., $x^2 + y^2 - 1 = 0$). Then differentiate equation to to find the derivative $y'(x_0)$ at the point

(x_0, y_0) of the graph (this derivative can also depend on y_0 but we do not write $y'(x_0, y_0)$ to make the notation simpler). Then the equation of the tangent line is

$$y = y'(x_0)(x - x_0) + y_0 = y'(x_0)x + (y_0 - y'(x_0)x).$$

- Preexam 1a, problems 6,7.
- Preexam 1b, problem 7.
- PSet 1, part II, problem 3.

3.2. Differentiable and continuous functions.

Functions given piecewise. We consider a function $f(x)$ given by $f(x) = \begin{cases} f_1(x), & x < x_0 \\ f_2(x), & x \geqslant x_0 \end{cases}$, where $f_1(x), f_2(x)$ are some differentiable functions. Then

(1) $f(x)$ is continuous if $f_1(x_0) = f_2(x_0)$.
(2) $f(x)$ is differentiable if it is continuous (i.e., $f_1(x_0) = f_2(x_0)$) and $f_1'(x_0) = f_2(x_0)$.

Often, $f_1(x)$ is fixed, and $f_2(x)$ depends on two parameters (say a, b), or vice versa. Then (1) typically leads to one equation on a, b, and (2) leads to two equations.

Problems to practice:

- Preexam 1a, problem 5.
- Preexam 1b, problem 5.

Continuouty/discontinuouty

Here problems ask to check whether given functions are continuous (or differentiable) and if not, where the points of discontinuouty are.

Problems to practice: Preexam 1a, problem 9.

3.3. Drawing graphs. *Problems to practice:*

- Suppl. notes 1D-4 (just some graphing).
- Wsh 1, problem 6 (graphs of inverses).