18.336 Problem Set 3

Due Thursday, 23 March 2006. This is the last problem set before the mid-term on Thurs. Apr. 6.

Problem 1: Staggered-grid Leap-frog

Take the two-component wave equation $u_t = bv_x - \sigma u$, $v_t = cu_x - \sigma u$, including the PML dissipation coefficient σ, where $b > 0$, $c > 0$, and $\sigma \geq 0$ are constants. Consider the staggered-grid leap-frog scheme:

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = b\frac{v_{m+1/2}^{n+1/2} - v_{m-1/2}^{n+1/2}}{\Delta x} - \sigma\frac{u_m^{n+1} + u_m^n}{2}$$

$$\frac{v_{m+1/2}^{n+3/2} - v_{m+1/2}^{n+1/2}}{\Delta t} = c\frac{u_{m+1}^{n+1} - u_m^{n+1}}{\Delta x} - \sigma\frac{v_{m+1/2}^{n+3/2} + v_{m+1/2}^{n+1/2}}{2}$$

(a) Apply a Von-Neumann analysis for $\sigma = 0$ to derive the CFL stability condition for this problem. You should get a 2×2 eigenproblem via the product of two 2×2 matrices, by writing:

$$\begin{pmatrix} \hat{u}^{n+1} \\ \hat{v}^{n+3/2} \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}\begin{pmatrix} \hat{u}^{n+1} \\ \hat{v}^{n+1/2} \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}\begin{pmatrix} \hat{u}^{n} \\ \hat{v}^{n+1/2} \end{pmatrix}$$

You needn't bother to analyze the defective case of two equal eigenvalues.

(b) Show that your CFL condition from (a) is a sufficient condition for stability with any $\sigma > 0$.

(c) For $\sigma = 0$, compute and plot the phase and group velocities (for several values of $\sqrt{bc}\lambda$) that you obtain in this scheme, by plugging in $u = e^{i(\theta m - \phi n)}$, $v = Ae^{i(\theta m - \phi n)}$, and solving for A and $\phi(\theta)$ ($\omega \Delta t = \phi$, $\beta \Delta x = \theta$).

Problem 2: PML, Matlab, and You

For this problem, you are going to implement the staggered leap-frog scheme, above, in Matlab, with periodic boundary conditions and PML boundary regions. Use the computational domain $x \in [0, 20]$ with $x = 0$ and $x = 20$ being equivalent. Use $b = c = 1$ everywhere, and $\sigma(x) \neq 0$ only in $[0, 1]$ (i.e. a PML region of thickness $L = 1$ at one end). Furthermore, use an initial condition $u_m^0 = v_m^{1/2} = 0$. Use $\Delta x = 0.1$ and $\lambda = 0.9$.

In order to start a wave moving, we will add a *source* term $s(x,t)$ to the u_t equation: $u_t = bv_x - \sigma u + s(x,t)$. In particular, you should use a Gaussian pulse at a single point $x = 10$:

$$s(x,t) = \delta(x - 10) \cdot e^{-(t-5)^2} \sin(5(t-5))$$

where to implement the $\delta(x-10)$ delta function in the discrete scheme, just add the source at a single m with amplitude multiplied by $1/\Delta x$. (Start all simulations at $t = 0$, i.e. assume $s(x, t < 0) = 0$.) This will produce *two* pulses starting at $x = 10$: one travelling right and one travelling left.For this problem, I used the pset2prob2.m file to implement the leap-frog scheme, where this file is posted on the web site.

(a) First, use $\sigma = 0$ everywhere. Compute where the *center* of the *right*-travelling $u(x,t)$ pulse is at $t = 10$. Now, predict where its center should be at $t = 510$ from your group-velocity calculation in problem 1. Compare this prediction to your simulation.

(b) Now, set σ to a *constant* σ_0 for $x < 1$ and $\sigma = 0$ otherwise. Predict the σ_0 that, for the *exact* PDE, would attenuate waves travelling through the PML by 10^{-4} in amplitude. Now, simulate it and find the actual factor by which the pulses are attenuated after they pass though and/or reflect from the PML, at $t = 30$.

(c) Now, set σ to a quadratic function $\sigma(x) = \sigma_2 x^2$ for $x \in [0, \frac{1}{2}]$, $\sigma(x) = \sigma_2(1-x)^2$ for $x \in (\frac{1}{2}, 1)$, and $\sigma = 0$ elsewhere. Again, predict the constant σ_2 so that in the exact PDE waves would be attenuated by 10^{-4}. Again compare this to what you actually get at $t = 30$.

Attach plots, etcetera, as necessary, in order to show the reader what you did.

Hint: store u_m^n and $v_{m+1/2}^{n+1/2}$ as two equal-length vectors `u(m)` and `v(m)` in Matlab, corresponding to $x = 0, \Delta x, \cdots, 20 - \Delta x$ and $x = \Delta x/2, 3\Delta x/2, \cdots, 20 - \Delta x/2$, respectively. The leap-frog update will then consist of two lines of Matlab code to update u and then v, where the space derivatives v_x and u_x are of the form `(v - [v(end),v(1:end-1)])/dx` and `([u(2:end),u(1)] - u)/dx` respectively.

Problem 3: Diffusion

Solve the diffusion/heat equation $u_t = bu_{xx}$ on $-1 \le x \le 1$ with periodic boundaries, $b = 1$, and initial data

$$u(x,0) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2} \\ \frac{1}{2} & \text{if } |x| = \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases}$$

Solve up to $t = \frac{1}{2}$. The exact solution is given by

$$u(x,t) = \frac{1}{2} + 2\sum_{\ell=0}^{\infty}(-1)^\ell \frac{\cos \pi(2\ell+1)x}{\pi(2\ell+1)} e^{-\pi^2(2\ell+1)^2 t}.$$

Use the Crank-Nicolson scheme

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = \frac{b}{2}\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{\Delta x^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}\right)$$

with $\Delta x = 0.05$.

(a) Compare the accuracy when $\mu = 1$ versus $\mu = 10$.

(b) Demonstrate that when $\lambda = \Delta x/\Delta t$ is constant, the error in the solution does not decrease with Δx when measured in the L_∞ norm, but it does decrease in the L_2 norm.